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Annuity, Bequests, Fertility and Longevity in Overlapping Generations Models

PhD thesis

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and in accordance with
the decision by the College of Deans.

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Contents

Chapter 1

Introduction

According to the United Nations' World Population Ageing Report (2015), the number of older persons - those aged 60 years or over - is expected to more than double by 2050 and to more than triple by 2100, rising from 962 million globally in 2017 to 2.1 billion in 2050. Globally, the population aged 60 or over is growing faster than all younger age groups. Managing old-age pension assets and insuring against longevity risk is one of major concerns for both governments and individuals in the 21st century.

In his seminal paper, Yaari (1965) showed that in the absence of any bequest motives, non-altruistic individuals should fully annuitize their assets to insure against the longevity risk (outliving their assets), provided that the annuity market is actuarially fair. Davidoff *et al.* (2005) demonstrated further that full annuitization is welfare-enhancing in a more general setting: when annuities are less than actuarially fair but provide a higher net rate of return than the capital market. Despite its theoretical attractiveness, *private* annuity markets are notoriously thin, and annuities are often over-priced and rarely purchased if they are available. The sharp contrast between the insurance function of annuities and almost non-existent private annuity markets are commonly dubbed 'Annuity Puzzle' (Inkmann *et al.* 2011).

Among the many explanations that individuals shy away from the annuity market, one is that annuities are priced unattractively due to *asymmetric information.* When individual's health is unobservable to annuity firms, *adverse selection* arises because healthy individuals are more likely to buy annuities. This implies the 'high-risk' types are overrepresented in the annuity purchasers. They crowd out unhealthy individuals and drive up the price of annuities. A traditional policy options to address adverse selection is to employ the social security system - a public annuitization tool - to include everyone in the annuity pool (see, for example, Eckstein *et al.*, 1985). These studies are often conducted in a partial equilibrium framework and the macroeconomic effects are missing. We explore further the details of addressing the origins of annuity puzzle and various policy options to enhance social welfare. For instance, we take into considerations the welfare implications of optimal pension rules and privatizing of social security. When individuals are heterogeneous in health status, the public social security system often redistributes resources from the unhealthy to the healthy. The redistribution role of social security system matters for the individual welfare and the optimal pension benefit rule needs to be arranged. When public social security does not necessarily enhance individual welfare in the general framework, the welfare implications of privatizing social security also needs to be investigated.

In Chapter 2 we build an overlapping-generations model to study the macroeconomic effects of private and public annuity markets with asymmetric information. We extend the work by Heijdra and Reijnders (2012) by assuming that individuals differ by two dimensions of heterogeneity - health and ability, which are positively correlated. We find that adverse selection caused by asymmetric information substantially reduces steady-state output per efficiency unit of labour and the capital intensity in the general equilibrium setting. The introduction of a social security system aggravates adverse selection and reduces output per efficiency unit and capital intensity further. The welfare effects of social security depend both on the individual type (health and ability) and the pension benefit rule. If pension benefits are proportional to an individual's contribution during youth and the percentage is fixed for everybody, then the social security system makes everybody worse off in the long run. Nevertheless, it is not a guarantee that the privatization of social security is Pareto improving for all generations. In the simulations we show that the abolition of a social security system featuring a proportional pension benefit rule will harm shock-time generations. That is, healthy individuals born at the time of the shock would have been better off if the social security system had not been privatized.

Chapter 3 gives rise to a possible explanation of the 'Annuity Puzzle'. Starting from Yaari (1965), economic theory would predict that at least a substantial share of individual assets should be annuitized, whilst in reality individuals hold their assets mainly in non-annuitized savings. Explanations for it have been sought both in the rational and the behavioral domain. Within the rational domain the focus has been predominantly on market imperfections such as asymmetric information and adverse selection. Another likely explanation is that people have a bequest motive, which substantially reduces their desire to annitize their assets. However, neither explanation alone could rationalize the annuity puzzle. As Davidoff *et al.* (2005) show in a two-period life-cycle model, individuals should annuitize all their assets as long as the annuity premium is higher than the return on capital markets, regardless of whether the annuity premium has been driven down by asymmetric information and adverse selection. And if the annuity market is actuarially-fair, individuals would be better off annuitizing all assets that they wish to consume in old age, despite a motive to leave bequest to their children. Hence, we combine the two commonly considered explanations of the annuity puzzle - asymmetric information and bequest motive - to show that their interplay can account for the 'Annuity Puzzle'.

The intuition behind the 'interplay mechanism' is that bequest motives enhance the value of non-annuitized savings so that in order for individuals to choose annuities over non-annuitized assets, they require annuities to priced nearly actuarially fair. This is, however, often not possible due to the asymmetric information and adverse selection. Later we extend the 'interplay mechanism' further by including a pay-as-you-go social security system and a health-wealth nexus. As to be expected, a public pension system aggravates the adverse selection and leads unhealthy individuals to retreat from the annuity market. And since social security is non-bequeathable, healthy individuals reduces the share of annuities in their retirement savings portfolio substantially. Similarly, a positive correlation between health and earning ability of individuals will aggravate adverse selection on the annuity market as the heavier annuity investment of healthier individuals will push down the annuity premium more than if such correlations are absent.

In Chapter 4 we investigate the welfare effects of opening up an annuity market in the presence of bequest motive in a general equilibrium context. While individuals may benefit from actuarially-fair - or at least not too unfair - annuities, it need not translate into social welfare improvement. As pointed out by previous studies, private and public benefits of annuities differ due to the loss of *unintended* bequests. Heijdra *et al.* (2014) show that the welfare effects can best be understood by reference to the Golden Rule. In the absence of annuities, unconsumed savings flow to the next generation, and this intergenerational transfer moves the economy closer to the Golden Rule as long as the economy is dynamically efficient. Opening up an annuity market cuts off this intergenerational transfer and as a consequence reduces capital intensity and individual welfare.

In this chapter we revisit the welfare implication of annuitization in the presence of *intended* bequests in a general equilibrium setting. By attaching a utility value to bequest, intended bequest motives can potentially mitigate the general equilibrium welfare loss by establishing the intergenerational transfer channel. Indeed, we find that stronger bequest motives are associated with higher capital accumulation because they entail a redistribution of assets from the older to the younger generation. We highlight the result that when bequest motives are taken into considerations, opening up an annuity market will lead to a decrease in capital accumulation and individual welfare (*Tragedy of Annuitization*). Meanwhile, we observe that this negative general equilibrium effect is dampened by the presence of bequest motives.

In Chapter 5 we develop a two-period overlapping-generations model where fertility is endogenous to study the optimal fertility rate for the society. A utility value is attached to fertility so that individuals derive utility not only from consumption, but also from their offsprings. Michel and Pestieau (1993) find that there exists an interior Samuelson Serendipity Equilibrium (SSE) for several constant elasticity of substitution (CES) utility and technology cases. We build on their work and relax the assumptions of full depreciation of capital, no utility derived from children, and no future labour supply. By using a numerical simulation we show that there exists an interior solution for the socially optimal fertility decision. However, the optimal fertility decision in the market equilibrium does not necessarily imply the socially optimum fertility decision. Thus, we compared two concepts of social optimization: the Samuelson Social Welfare concept which maximizes the steady-state welfare of a representative young individual, and Social Welfare Function where the social optimum is dynamically consistent. We prefer the latter one since it provides us a tool to analyze individual decisions across generations. Finally, we show that a market economy with child taxes and intergenerational transfers replicates the first-best social optimum under the social welfare function. Last but not least, we find that the transitional path to the social optimum is an improvement in the long run.

This thesis aims to study the financial and social insurance against longevity risk both from the individual's perspective and the society as a whole. Interestingly, utility-maximizing individual decisions do not always lead to the socially optimal equilibrium. The mechanisms behind are better-understood in our small macroeconomic models. We aim to provide an insight into the roles of annuity, bequest and fertility in providing insurance against longevity risk for the old-age people. We hope that this thesis contributes to the discussion of old-age insurance when ageing has become one of the most urging problems globally in the 21st century.

Chapter 2

The Macroeconomic effects of longevity risk under private and public insurance and asymmetric information*

*This chapter is based on Heijdra, Jiang and Mierau (2019a).

2.1 Introduction

More than half a century ago Yaari (1965) proved convincingly that private annuities are very attractive insurance instruments when non-altruistic individuals face longevity risk. Simply put, annuities are desirable because they insure such agents against the risk of outliving their assets. Yaari also proved a much stronger result: in the absence of an intentional bequest motive, rational utility-maximizing individuals should *fully* annuitize all of their savings. Yaari derives this result under the strong assumption that actuarially fair annuities are available. In a more recent paper, however, Davidoff *et al.* (2005) have demonstrated that the full annuitization result holds in a much more general setting than the one adopted by Yaari, for example when annuities are less than actuarially fair.

Despite the theoretical attractiveness of annuities, there is a vast body of empirical evidence showing that in reality people do not invest heavily in private annuity markets. The discrepancy between the theoretical predictions and the observable facts regarding annuity markets is known as the annuity puzzle. Of course there are many reasons why individuals may not choose to fully annuitize their wealth. Friedman and Warshawsky (1990, pp. 136-7), for example, argue that purchases of private annuities are low because (a) individuals may want to leave bequests to their offspring, (b) agents may already implicitly hold social annuities because they are participating in a system of mandatory public pensions, and (c) private annuities may be priced unattractively, for example because of transaction costs and taxes, excessive profits extracted by imperfectly competitive annuity firms, and adverse selection. Intuitively, under asymmetric information annuity companies cannot observe an individual's health status. Adverse selection arises in such a setting because agents with above-average health are more likely to buy annuities. This implies that such "high-risk types" are overrepresented in the group of clients of annuity firms and that pricing of annuities cannot be based on the average health status of the population at large.

While recognizing their potential role in accounting for parts of the annuity puzzle, we ignore intentional bequest motives, administrative costs, and imperfect competition in this Chapter.[1] Instead, we follow *inter alia* Abel (1986), Walliser (2000), Palmon and Spivak (2007), Sheshinski (2008), and Heijdra and Reijnders (2012) by focusing on the adverse selection channel. We approach the material sequentially by first demonstrating the adverse selection effect in an economy without public pensions. In the next step we introduce social annuities and study the general equilibrium interactions between private and public annuity markets under different pension benefit rules.

This Chapter is most closely related to earlier work by Heijdra and Reijnders (2012). They study a discrete-time overlapping generations model in which non-altruistic agents differ in their innate health status, which is assumed to be private information. The private annuity market settles in a risk-pooling equilibrium in

[1] These features will be studied in depth in Chapter 3 and 4.

which the unhealthiest segment of the population experiences binding borrowing constraints (because they are unable to go short on annuities) and the other agents receive a common yield on their annuity purchases. They also show that the introduction of a mandatory public pension system—though immune to adverse selection by design—leads to a reduction in steady-state welfare, an aggravation of adverse selection in the private annuity market, and a reduction in the economy-wide capital intensity.

We extend the work by Heijdra and Reijnders (2012) by assuming that the individuals populating the economy differ by *two dimensions* of heterogeneity (health and ability) rather than just a single one (health). The introduction of heterogeneous abilities serves two purposes. First, as was shown by Walliser (2000, pp. 374-375) in a partial equilibrium setting, "(the simulations reveal that) between 40 and 60 percent of the measured adverse selection is due to the positive correlation between income and mortality…" By incorporating health-ability heterogeneity, and by assuming that there is a positive correlation between the two characteristics, we are able to capture this reputedly important source of adverse selection in the private annuity market. There is a second reason why heterogeneity matters which is related to the type of funded public pension system that is in place. Indeed, depending on the details regarding pension contributions and receipts, social security systems can have vastly different welfare implications for consumers with different health status and/or ability. In this paper we consider three different public pension schemes which differ in the degree to which they lead to (implicit or explicit) redistribution from healthy to unhealthy individuals.

Our main findings are as follows. Firstly, a plausibly calibrated version of the model reveals that, compared to the case with full information, asymmetric information on the part of annuity companies is important quantitatively in that it causes substantial reductions in steady-state output per efficiency unit of labour and the capital intensity. The general equilibrium effects are thus shown to matter a lot. Second, the introduction of a funded social security system reduces the capital intensity and output per efficiency unit even further, more so the larger is the system, i.e. the higher is the replacement rate it incorporates. These results are consistent with Palmon and Spivak (2007) and Heijdra and Reijnders (2012). Third, privatizing social security (by abolishing the public pension system) is not generally Pareto improving to *all* generations. Indeed, in our simulations we find that healthy agents born at the time of the shock would have been better off if the social security system had not been privatized. Just as for unfunded pensions, getting rid of a pre-existing funded system is not an easy task to accomplish.

The remainder of the Chapter is organized as follows. In Section 2.2 we set up the model and characterize the microeconomic choices and the resulting macroeconomic equilibrium under full information, i.e. the hypothetical case in which insurance companies can perfectly observe an individual's characteristics. In Section 2.3 we introduce asymmetric information inhibiting insurance companies and show that it leads to a pooling equilibrium in the annuity market. In Section 2.4 we introduce a fully-funded social security system in which pension contributions are proportional to labour income during youth. We analyze three specific versions of this system which differ with respect to the pension receipts during old age. Section 2.5 considers the consequences of privatizing social security. The final section concludes. Some technical issues are dealt with in three brief appendices.

2.2 Model

2.2.1 Consumers

In each period the population in the closed economy under consideration features two overlapping generations of heterogeneous agents. Each person can live at most for two periods, namely 'youth' (superscript y) and 'old age' (superscript o). Individuals are heterogenous along two exogenously given dimensions. First, they differ by health status which we capture by the probability of surviving into old-age. Everyone faces lifetime uncertainty at the end of the first period, and the survival probability is denoted by μ. This means that unhealthy people have a higher risk of dying and a shorter expected life span (which equals $1+\mu$ periods). Second, individuals differ in their working ability as proxied by innate labour productivity η.

We assume that consumer types are continuous and uniformly distributed on these two dimensions, i.e. $\mu \in [\mu_L, \mu_H]$ (such that $0 < \mu_L < \mu_H < 1$) and $\eta \in [\eta_L, \eta_H]$ (such that $0 < \eta_L < \eta_H$). Furthermore, we postulate that μ and η are positively correlated. Hence, a person in better health is more likely to possess higher working abilities, and vice versa. The bivariate uniform distribution used in this paper is characterized by the following probability density function:

$$h(\mu, \eta) = \frac{1 + \xi\,(\mu - \bar{\mu})(\eta - \bar{\eta})}{(\mu_H - \mu_L)(\eta_H - \eta_L)}, \tag{2.1}$$

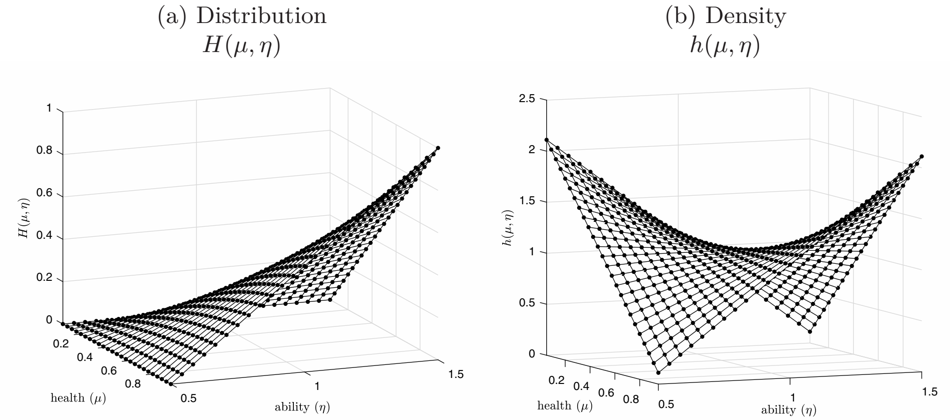
where ξ is a parameter regulating the correlation between μ and η (such that $\xi > 0$), and $\bar{\mu}$ and $\bar{\eta}$ denote the unconditional means of μ and η, respectively. In Figure 2.1 the distribution function is depicted in panel (a) whilst the probability density function is shown in panel (b). From the graph of the density function it is clear that there is a higher probability for healthier consumers to possess higher working abilities, and vice versa. For future reference we postulate Lemma 1 which summarizes some useful properties of the bivariate distribution that we employ.

Lemma 1. *The distribution function for the survival probability μ and labour productivity η is given by:*

$$H(\mu, \eta) = \frac{(\mu - \mu_L)(\eta - \eta_L)}{(\mu_H - \mu_L)(\eta_H - \eta_L)} \left[1 + \frac{\xi}{4}\,(\mu_H - \mu)(\eta_H - \eta)\right],$$

where $\mu_L \leq \mu \leq \mu_H$ and $\eta_L \leq \eta \leq \eta_H$. The density function is given in (2.1). Further properties of the distribution are: (i) the marginal density functions are $h_\mu(\mu) = 1/(\mu_H - \mu_L)$ and $h_\eta(\mu) = 1/(\eta_H - \eta_L)$; (ii) the unconditional means are

Figure 2.1: Features of the distribution for μ and η



Legend Health and innate ability are proxied by, respectively, the survival probability μ and the labour productvity parameter η. The two characteristics of an individual are positively correlated. The distribution $H(\mu, \eta)$ is bivariate uniform. The marginal distributions of μ and η are both uniform. See Appendix A and Lemma 2.1 for further features of the distribution.

$\bar{\mu} = (\mu_L + \mu_H)/2$ and $\bar{\eta} = (\eta_L + \eta_H)/2$; (iii) the unconditional variances are $\sigma_\mu^2 = (\mu_H - \mu_L)^2/12$ and $\sigma_\eta^2 = (\eta_H - \eta_L)^2/12$; (iv) the covariance is $\operatorname{cov}(\eta, \mu) = \xi\sigma_\eta^2\sigma_\mu^2$ and the correlation is $\operatorname{cor}(\eta, \mu) = \xi\sigma_\eta\sigma_\mu$; (v) the conditional probability density functions are:

$$h_{\mu|\eta}(\mu) \equiv \frac{h(\eta, \mu)}{h_\eta(\eta)} = \frac{1 + \xi(\mu - \bar{\mu})(\eta - \bar{\eta})}{\mu_H - \mu_L},$$
$$h_{\eta|\mu}(\eta) \equiv \frac{h(\eta, \mu)}{h_\mu(\mu)} = \frac{1 + \xi(\mu - \bar{\mu})(\eta - \bar{\eta})}{\eta_H - \eta_L},$$

and (vi) the conditional mean of η for a given μ is:

$$\Gamma_1(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \eta h(\eta, \mu)\, d\eta}{\int_{\eta_L}^{\eta_H} h(\eta, \mu)\, d\eta} = \frac{h_\mu(\mu) \int_{\eta_L}^{\eta_H} \eta h_{\eta|\mu}(\eta)\, d\eta}{h_\mu(\mu)} = \bar{\eta} + \xi\sigma_\eta^2(\mu - \bar{\mu}).$$

***Proof**: see Appendix A.* ■

From the perspective of birth, the expected lifetime utility of a person with health status μ and working ability η is given by:

$$\mathbb{E}\Lambda_t(\mu, \eta) \equiv U\left(C_t^y(\mu, \eta)\right) + \mu\beta U\left(C_{t+1}^o(\mu, \eta)\right), \tag{2.2}$$

where $C_t^y(\mu, \eta)$ and $C_{t+1}^o(\mu, \eta)$ are consumption during youth and old age, respectively, β is the parameter capturing pure time preference $(0 < \beta < 1)$, and $U(C)$ is the felicity function:

$$U(C) \equiv \begin{cases} \dfrac{C^{1-1/\sigma}-1}{1-1/\sigma}, & \text{for } \sigma \neq 1, \\ \ln C & \text{for } \sigma = 1, \end{cases} \tag{2.3}$$

where σ is the intertemporal elasticity of substitution ($\sigma > 0$). Equation (2.2) incorporates the assumption that individuals do not have a bequest motive, i.e. utility solely depends on own consumption during one's lifetime.

In this section we postulate the existence of perfect private annuities. Specifically, we adopt the following assumptions regarding the market for private annuities:

(A0) Health status is public information.

(A1) The annuity market is perfectly competitive. A large number of risk-neutral firms offer annuities to individuals, and annuity firms can freely enter or exit the market.

(A2) Annuity firms do not use up any real resources.

As is explained by Heijdra and Reijnders (2012, pp. 322–3), in this *Full Information* case (abbreviated as FI) each health type receives its actuarially fair rate of return and achieves perfect insurance against longevity risk. If $A_t^p(\mu,\eta)$ denotes the private annuity holdings of an agent of health type μ then the net rate of return on annuities will be equal to:

$$1 + r_{t+1}^p(\mu) = \frac{1 + r_{t+1}}{\mu}, \tag{2.4}$$

where r_{t+1} is the net rate of return on physical capital (see also below). Since the survival rate is such that $0 < \mu < 1$, it follows from (2.4) that $r_{t+1}^p(\mu)$ exceeds r_{t+1} so that all agents will completely annuitize their wealth. This classic result was first derived by Yaari (1965).

We assume that individuals work full time during youth and part time in old age as a result of a system of mandatory retirement. With full annuitization of assets the periodic budget identities are given by:

$$C_t^y(\mu,\eta) + A_t^p(\mu,\eta) = w_t(\eta), \tag{2.5}$$

$$C_{t+1}^o(\mu,\eta) = \lambda w_{t+1}(\eta) + (1 + r_{t+1}^p(\mu))A_t^p(\mu,\eta), \tag{2.6}$$

where $w_t(\eta)$ is the wage rate of an η type in period t, λ is the proportion of time that is devoted to work in old age ($0 < \lambda < 1$), and $1 + r_{t+1}^p(\mu)$ is the rate of return on private annuities. The periodic budget identities can be combined to obtain the consolidated budget constraint:

$$C_t^y(\mu,\eta) + \frac{C_{t+1}^o(\mu,\eta)}{1 + r_{t+1}^p(\mu)} = w_t(\eta) + \frac{\lambda w_{t+1}(\eta)}{1 + r_{t+1}^p(\mu)}. \tag{2.7}$$

The present value of lifetime consumption (left-hand side) equals the present value of lifetime income (right-hand side). That is, people consume their human wealth.

Consumers choose $C_t^y(\mu,\eta)$ and $C_{t+1}^o(\mu,\eta)$ in order to maximize expected lifetime utility (2.2) subject to the budget constraint (2.7). The optimal consumption plans and annuity demands are fully characterized by:

$$C_t^y(\mu,\eta) = \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\left[w_t(\eta) + \frac{\lambda\mu w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.8}$$

$$\frac{\mu C_{t+1}^o(\mu,\eta)}{1+r_{t+1}} = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\right]\left[w_t(\eta) + \frac{\lambda\mu w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.9}$$

$$A_t^p(\mu,\eta) = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\right]w_t(\eta) - \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\frac{\lambda\mu w_{t+1}(\eta)}{1+r_{t+1}}, \tag{2.10}$$

where we have substituted the expression for the actuarially fair annuity rate (2.4), and where $\Phi(\mu,x)$ is the marginal propensity to consume out of lifetime income during youth:

$$\Phi(\mu,x) \equiv \frac{1}{1+(\mu\beta)^\sigma x^{\sigma-1}}. \tag{2.11}$$

From equations (2.8) and (2.9) we find that consumption during youth and old-age are both proportional to human wealth. Furthermore, equation (2.10) shows that annuity demand depends positively on the wage income during youth and negatively on old-age labour income.

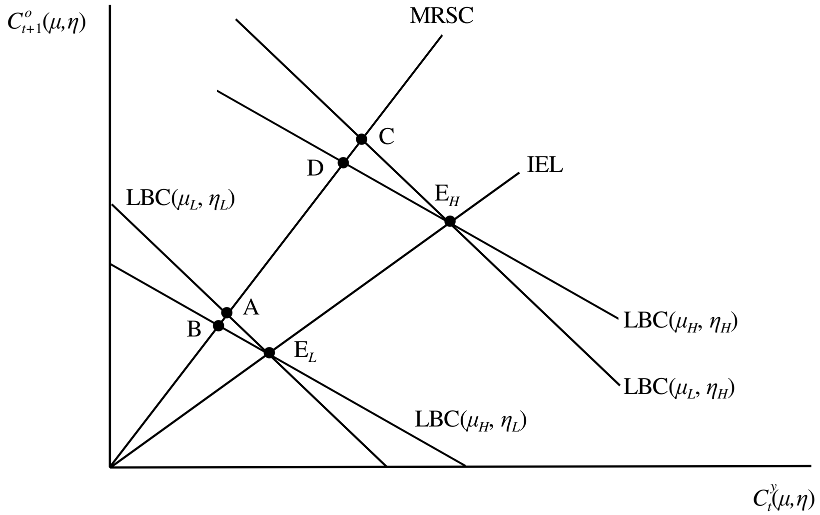
The optimal consumption choices of different types of consumers are illustrated in Figure 2.2. To avoid cluttering the diagram we illustrate the choices made by the four extreme types, unhealthy and healthy lowest-skilled (μ_L,η_L) and (μ_H,η_L), and unhealthy and healthy highest-skilled (μ_L,η_H) and (μ_H,η_H). For a given working ability type η_i, the line labelled LBC(μ_L,η_i) and LBC(μ_H,η_i) are the lifetime budget constraints as given in (2.7). For skill type η_L the income endowment point $(w_t(\eta),\lambda w_{t+1}(\eta))$ is located at point E_L. With perfect annuities, LBC(μ_L,η_i) is steeper than LBC(μ_H,η_i) because the unhealthy get a much higher annuity rate than the healthy.

In the presence of perfect annuities and under full annuitization, the consumption Euler equation is given by:

$$\frac{U'(C_t^y(\mu,\eta))}{\beta U'(C_{t+1}^o(\mu,\eta))} = \mu\left(1+r_{t+1}^p(\mu)\right) = 1+r_{t+1}, \tag{2.12}$$

where we have used (2.4) to get from the first to the second equality. The crucial thing to note is that all agents equate the marginal rate of substitution between current and future consumption to the gross interest factor on capital. Intuitively, as was first pointed out by Yaari (1965), the mortality rate drops out of the expression characterizing the life-cycle profile of consumption because agents are fully insured against the unpleasant aspects of lifetime uncertainty. For the homothetic felicity function (2.3) it is easy to show that (2.12) is a ray from the origin—see the locus labelled MRSC in Figure 2.2. Optimal choices are located at the intersection of

Figure 2.2: Consumption-saving choices under full information



Legend LBC(μ_i, η_j) is the lifetime budget constraint for an individual with survival probability μ_i and productivity level η_j. IEL is the income endowment line and agents are located on the line segment $\mathrm{E}_L\mathrm{E}_H$. MRSC is the consumption Euler equation under perfect information with actuarially fair annuities at the individual level. Optimal consumption for individual (μ_i, η_j) is located at the intersection of MRSC and LBC(μ_i, η_j). All individuals purchase annuities.

MRSC and the relevant lifetime budget constraint. It follows that types (μ_L, η_L) and (μ_H, η_L) consume at points A and B respectively.

What about the choices made by the highest-ability types? Given the specification of technology adopted below, it follows that $w_t(\eta) = \eta w_t$ and $w_{t+1}(\eta) = \eta w_{t+1}$ so that income endowment points lie along the ray from the origin labelled IEL. Furthermore, it follows from (2.7) that LBC(μ_L, η_H) is parallel to LBC(μ_L, η_L) whilst LBC(μ_H, η_H) is parallel to LBC(μ_H, η_L). Hence types (μ_L, η_H) and (μ_H, η_H) consume at points C and D respectively.

Several conclusions can be drawn from the microeconomic behaviour discussed in this subsection. First, in this closed economy featuring a positive capital stock (see below) all agents are net savers, i.e. everybody expresses a positive demand for private annuities, $A_t^p(\mu, \eta) > 0$ for all μ and η. This result follows readily from Figure 2.2 because the MRSC line lies to the left of the IEL line. Second, for a given value of agent productivity η, the demand for annuities is increasing in the survival probability μ, i.e. $\partial A_t^p(\mu, \eta)/\partial\mu > 0$. Intuitively, healthy people buy more annuities than do unhealthy people of the same skill category because they expect to live longer a priori. Again this result follows readily from Figure 2.2 because LBC(μ_L, η_i) is steeper than LBC(μ_H, η_i). Third, the demand for annuities is increasing in the skill level, i.e. $\partial A_t^p(\mu, \eta)/\partial\eta > 0$. This can be see graphically in Figure 2.2 and can be proved formally by noting that $A_t^p(\mu, \eta)$ in (2.10) is linear in η.

2.2.2 Demography

Let L_t denote the size of the population cohort born at time t. The density of consumers with health type μ and working ability η is thus:

$$L_t(\mu, \eta) \equiv h(\mu, \eta) L_t, \tag{2.13}$$

where the density function $h(\mu, \eta)$ is stated in (2.1) above. The density of (young and old) consumers of type μ alive at time t is given by:

$$P_t(\mu) \equiv \mu \int_{\eta_L}^{\eta_H} L_{t-1}(\mu, \eta) d\eta + \int_{\eta_L}^{\eta_H} L_t(\mu, \eta) d\eta = \mu h_\mu(\mu) L_{t-1} + h_\mu(\mu) L_t, \tag{2.14}$$

where $h_\mu(\mu)$ is the marginal distribution of μ (see Lemma 1(i)). If newborn cohort sizes evolves according to $L_t = (1+n)L_{t-1}$ (with $n > -1$), the total population at time t is given by:

$$P_t \equiv \int_{\mu_L}^{\mu_H} P_t(\mu) d\mu = \frac{1+n+\bar{\mu}}{1+n} L_t, \tag{2.15}$$

where $\bar{\mu} \equiv \int_{\mu_L}^{\mu_H} \mu h_\mu(\mu) d\mu$ is the average survival rate of a newborn cohort.

2.2.3 Production

We assume that perfect competition prevails in the goods market. The technology is represented by the following Cobb-Douglas production function:

$$Y_t = \Omega_0 K_t^{\varepsilon} N_t^{1-\varepsilon}, \tag{2.16}$$

where Y_t is total production, K_t is the aggregate capital stock, ε is the efficiency parameter of capital ($0 < \varepsilon < 1$), Ω_0 is total factor productivity (assumed to be constant), and N_t is the *effective* labor force, which is defined as:

$$N_t \equiv \int_{\eta_L}^{\eta_H} \int_{\mu_L}^{\mu_H} \eta \left[L_t(\mu, \eta) + \lambda L_{t-1}(\mu, \eta) \right] d\mu d\eta. \tag{2.17}$$

Note that N_t has the dimension of worker efficiency (denoted by η) times number of working hours. By using (2.13) in (2.17) and noting that $L_t = (1+n)L_{t-1}$ we find that N_t/L_t can be written as:

$$\frac{N_t}{L_t} = \bar{\eta} + \frac{\lambda}{1+n} \left[\bar{\eta}\bar{\mu} + \mathrm{cov}\,(\eta, \mu) \right], \tag{2.18}$$

where $\mathrm{cov}\,(\eta, \mu) \equiv \xi \sigma_\eta^2 \sigma_\mu^2$ is the (positive) covariance between μ and η (see Lemma 1(iv)).

By defining $y_t \equiv Y_t/N_t$ and $k_t \equiv K_t/N_t$, the intensive-form production function can be written as:

$$y_t = \Omega_0 k_t^{\varepsilon}. \tag{2.19}$$

Firms choose efficiency units of labour and the capital stock such that profits are maximized. This optimization problem gives the following factor demand equations:

$$r_t + \delta = \varepsilon \Omega_0 k_t^{\varepsilon - 1}, \tag{2.20}$$

$$w_t = (1-\varepsilon)\Omega_0 k_t^{\varepsilon}, \tag{2.21}$$

$$w_t(\eta) = \eta w_t, \tag{2.22}$$

where r_t is the net rate of return on physical capital, δ is the depreciation rate of capital ($0 < \delta < 1$), and w_t is the rental rate on efficiency units of labour. With perfect substitutability of efficiency units of labour, the wage rate of a η type worker, $w_t(\eta)$, is η times the rental rate w_t (as was asserted above).

2.2.4 Equilibrium

The model is completed by a description of the macroeconomic equilibrium. Since all annuity purchases are invested in the capital market we find that:

$$K_{t+1} = L_t \int_{\mu_L}^{\mu_H} \int_{\eta_L}^{\eta_H} A_t^p(\mu, \eta) h(\mu, \eta) d\eta d\mu, \tag{2.23}$$

where $A_t^p(\mu, \eta)$ is given in (2.10) above. Intuitively, equation (2.23) says that next period's aggregate capital stock is equal to total savings in the current period (consisting of private annuities). By substituting the demand for annuities (2.10) and the wage equation (2.22) into (2.23) we obtain the fundamental difference equation for the capital intensity:

$$k_{t+1} = \frac{1}{1+n} \frac{L_t}{N_t} \left[\bar{\eta} w_t - \int_{\mu_L}^{\mu_H} \Phi\left(\mu, \frac{1+r_{t+1}}{\mu} \right) \left[w_t + \frac{\lambda \mu w_{t+1}}{1+r_{t+1}} \right] h_\mu(\mu) \Gamma_1(\mu) d\mu \right], \tag{2.24}$$

where $\Gamma_1(\mu)$ is the conditional mean of η given μ (see Lemma 1(vi)). In view of (2.20)–(2.21) w_t and r_{t+1} depend on, respectively, k_t and k_{t+1} so (2.24) is a non-linear implicit function relating k_{t+1} to k_t and the exogenous variables.

2.2.5 Parameterization and visualization

In order to visualize the main features of the economy we parameterize the model by selecting plausible values for the structural parameters—see Table 2.1. We follow Heijdra and Reijnders (2012) in the parameterization procedure. First, we postulate plausible values for the intertemporal elasticity of substitution ($\sigma = 0.7$), the efficiency parameter of capital ($\varepsilon = 0.275$), the annual capital depreciation rate ($\delta_a = 0.06$), the annual growth rate of the population ($n_a = 0.01$) and the target annual steady-state interest rate ($\hat{r}_a = 0.05$). Using these parameters we can determine the steady-state (annual) capital-output ratio ($\hat{K}/\hat{Y} = \varepsilon/(\hat{r}_a + \delta_a) = 2.5$). Second, we set the length of each period to be 40 years and compute the values for n, δ and $\hat{r}$ (noting that $n = (1+n_a)^{40} - 1$, $\delta = 1-(1-\delta_a)^{40}$ and $\hat{r} = (1+r_a)^{40} - 1$). Third, we assume that the mandatory retirement age is 65 years so that $\lambda = 25/40 = 0.625$. In the fourth step, we choose $\eta_L = 0.5$, $\eta_H = 1.5$, $\mu_L = 0.05$, $\mu_H = 0.95$, so that the average health status is $\bar{\mu} = 0.5$, average working ability is $\bar{\eta} = 1$, and the variances are $\sigma_\eta^2 = 0.0833$ and $\sigma_\eta^2 = 0.0675$. By setting $\xi = 4$ we ensure that there is a strong correlation between health and ability, i.e. $\text{cor}(\mu, \eta) = 0.300$.[2] In the fifth step we choose Ω_0 such that $\hat{y} = 10$ in the initial steady state. This also pins down the steady state values for $\hat{k}$ and $\hat{w}$. In the final step the discount factor β is used as a calibration parameter, i.e. it is set at the value such that the steady-state version of the fundamental difference equation (2.24) is satisfied. To interpret the value of β in Table 2.1, note that the annual rate of time preference is $\rho_a = \beta^{-1/40} - 1 = 0.0204$ (a little over two percent per annum).

The main features of the steady-state FI equilibrium are reported in column (a) of Table 2.2. Consistent with the calibration procedure, output per efficiency unit of labour is equal to ten ($\hat{y} = 10$) whilst the steady-state interest rate is five percent on an annual basis ($\hat{r}^a = 0.05$). The steady-state capital intensity equals $\hat{k} = 0.395$. Ownership of the capital stock is highly uneven due to the fact that individuals differ in terms of labour productivity. Indeed, as is noted in the table, the first ability quartile of agents (averaged over all survival rates) owns 12.34% of the capital stock. In contrast, the top ability quartile owns 39.12% of the economy's stock of capital.

Steady-state consumption (per efficiency unit of labour) by the young and sur-

[2]The positive correlation between health and income (or productivity) is mentioned by many authors in the literature on annuities—see, for example, Walliser (2000), Brunner and Pech (2008), Direr (2010), and Cremer *et al.* (2010). Firm empirical evidence on this correlation is, however, hard to come by. In a recent paper Chetty *et al.* (2016) employ US data for the period 2001-2014 and find that the gap in life expectancy between the richest and poorest 1% of individuals was 14.6 years for men and 10.1 years for women. In our calibration the expected lifetime at birth of the bottom and top 1% individuals (by productivity) are 54.65 and 65.35.

Table 2.1: Structural parameters

σ	intertemporal substitution elasticity		0.7000
ε	capital efficiency parameter		0.2750
δ_a	annual capital depreciation rate		0.0600
δ	capital depreciation factor		0.9158
n_a	population growth rate		0.0100
n	population growth factor		0.4889
β	time preference parameter	c	0.4462
λ	mandatory retirement parameter		0.6250
Ω_0	scale factor production function	c	12.9071
μ_L	survival rate of the unhealthiest		0.0500
μ_H	survival rate of the healthiest		0.9500
η_L	lowest working ability		0.5000
η_H	highest working ability		1.5000
ξ	covariance parameter of the distribution function		4.0000

Note The parameters labelled 'c' are calibrated as is explained in the text. The remaining parameters are postulated a priori. The values for δ and n follow from, respectively, δ_a and n_a, by noting that each model period represents 40 years.

viving old are given by:

$$\hat{c}^y \equiv \frac{L_t}{N_t} \int_{\eta_L}^{\eta_H} \int_{\mu_L}^{\mu_H} \hat{C}^y(\mu, \eta)\, h(\mu, \eta) d\mu d\eta, \tag{2.25}$$

$$\hat{c}^o \equiv \frac{1}{1+n} \frac{L_t}{N_t} \left[\int_{\eta_L}^{\eta_H} \int_{\mu_l}^{\mu_h} \mu \hat{C}^o(\mu, \eta)\, h(\mu, \eta) d\mu d\eta \right]. \tag{2.26}$$

Inequality due to heterogeneous productivity also shows up in the consumption levels during youth and old-age. The two lowest-ability quartiles enjoy a modest and declining share of total consumption over the life-cycle due to the positive correlation between health and ability. The opposite holds for the two highest-ability quartiles. Finally, Table 2.2 also reports some welfare indicators. Not surprisingly we find that expected lifetime utility is lowest for individuals with low ability and poor health (μ_L, η_L) and highest for those lucky ones with high ability and excellent health (μ_H, η_H).[3]

In Figure 2.3 we depict the steady-state profiles for youth consumption, old-age consumption, annuity demand, and expected utility. These profiles have been

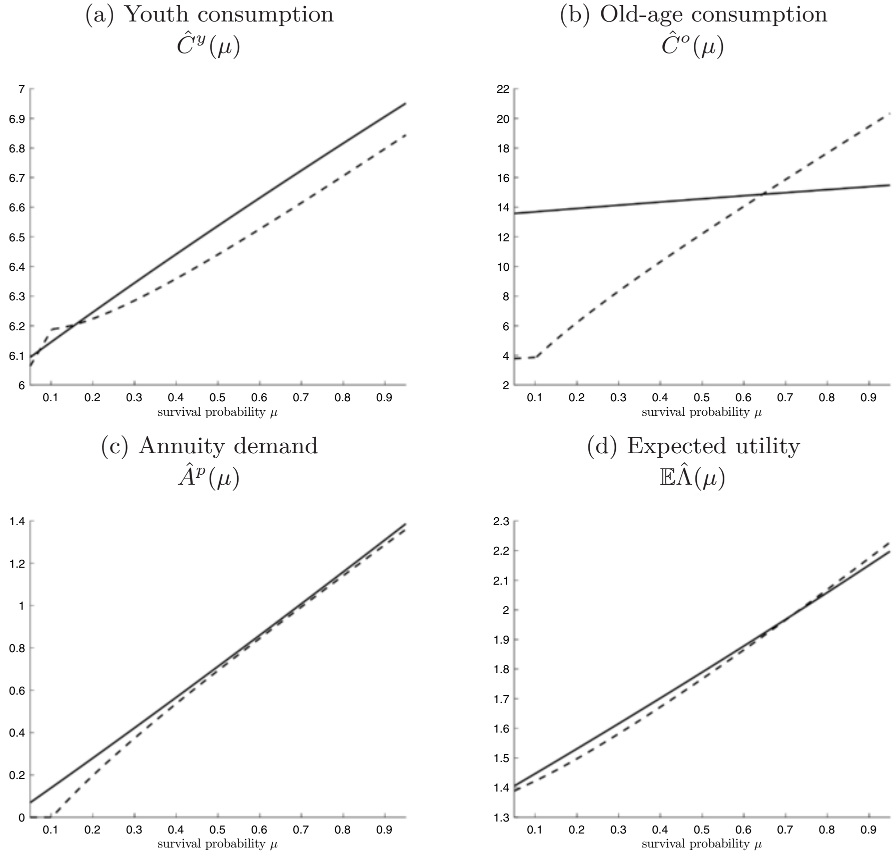
[3] By scaling steady-state output such that $\hat{y} = 10$ for the FI case we avoid the counterintuitive feature noted by Heijdra and Reijnders (2012, p. 321) that lifetime utility is decreasing in the survival probability.

Table 2.2: Allocation and welfare

	(a) FI	(b) AI	(c) SA_A $\theta = 0.010$	(d) SA_A $\theta = 0.025$	(e) SA_B $\theta = 0.010$	(f) SA_B $\theta = 0.025$	(g) SA_C $\theta = 0.010$	(h) SA_C $\theta = 0.025$
$\hat{y}$	10.000	9.840	9.776	9.680	9.768	9.668	9.762	9.660
$\hat{k}$	0.395	0.373	0.364	0.351	0.363	0.350	0.362	0.349
%Q1	12.34	11.78	10.15	7.73	9.69	6.69	9.15	5.50
%Q2	19.81	19.46	17.14	13.59	16.90	12.98	16.75	12.60
%Q3	28.73	28.84	25.81	21.05	25.92	21.26	26.12	21.66
%Q4	39.12	39.93	36.18	30.11	36.74	31.46	37.22	32.58
%SAS			10.72	27.51	10.74	27.60	10.76	27.67
$\hat{r}$	6.04	6.34	6.47	6.66	6.48	6.69	6.50	6.70
$\hat{r}^a$	5.00%	5.11%	5.16%	5.22%	5.16%	5.23%	5.17%	5.24%
$\hat{w}$	7.250	7.134	7.087	7.018	7.082	7.010	7.077	7.003
$\widehat{BC}$	0.00%	5.83%	10.03%	17.66%	10.63%	19.33%	10.63%	19.33%
$\hat{r}^p$		10.18	10.12	9.99	10.12	9.98	10.12	9.96
$\hat{\mu}^p$		0.66	0.67	0.70	0.67	0.70	0.67	0.70
$\widehat{AS}$		1.31	1.34	1.39	1.35	1.40	1.35	1.41
$\hat{c}^y$	5.357	5.296	5.270	5.233	5.268	5.228	5.265	5.225
%Q1	15.99	16.03	16.02	15.98	16.06	16.09	16.12	16.20
%Q2	22.10	22.13	22.12	22.10	22.14	22.16	22.16	22.20
%Q3	28.06	28.05	28.05	28.06	28.04	28.04	28.02	27.99
%Q4	33.85	33.79	33.81	33.86	33.75	33.72	33.70	33.61
$\hat{c}^o$	4.087	4.021	3.994	3.954	3.991	3.949	3.988	3.946
%Q1	12.23	10.70	10.72	10.77	10.77	10.93	10.83	11.14
%Q2	19.74	18.78	18.79	18.82	18.82	18.90	18.83	18.95
%Q3	28.75	29.04	29.03	29.02	29.02	28.98	29.00	28.91
%Q4	39.28	41.48	41.46	41.39	41.39	41.18	41.33	41.00
$\mathbb{E}\hat{\Lambda}(\mu_L, \eta_L)$	1.014	0.996	0.989	0.978	1.022	1.020	1.026	1.029
$\mathbb{E}\hat{\Lambda}(\mu_H, \eta_L)$	1.433	1.471	1.468	1.463	1.260	1.261	1.266	1.276
$\mathbb{E}\hat{\Lambda}(\mu_L, \eta_H)$	1.529	1.517	1.513	1.506	1.532	1.527	1.531	1.525
$\mathbb{E}\hat{\Lambda}(\mu_H, \eta_H)$	2.143	2.167	2.164	2.161	2.031	2.026	2.030	2.024

Note Here %Qj denotes the share accounted for by skill quartile j (averaged over all survival rates) of the variable directly above it. %SAS is the share owned by the social annuity system. $\mathbb{E}\hat{\Lambda}(\mu_i, \eta_j)$ gives expected utility for an agent with health type μ_i and skill type η_i. $\widehat{BC}$ is the proportion of the population facing borrowing constraints. $\widehat{AS}$ is an indicator for the severity of adverse selection in the private annuity market.

Figure 2.3: Steady-state profiles



Legend The solid lines depict the steady-state profiles for the full information (FI) case featuring perfect annuities. The dashed lines visualize the profiles for the asymmetric information (AI) case in which adverse selection results in a single pooling rate of interest on annuities, $\bar{r}^p_{t+1}$. In the AI case agents with poor health face binding borrowing constraints regardless of their productivity in the labour market.

averaged over η values and are thus a function of the survival probability only:

$$\hat{C}^y(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \hat{C}^y(\mu,\eta)\, h(\mu,\eta) d\eta}{\int_{\eta_L}^{\eta_H} h(\mu,\eta) d\eta} = \Phi\left(\mu, \frac{1+\hat{r}}{\mu}\right)\left[1+\frac{\lambda\mu}{1+\hat{r}}\right]\hat{w}\Gamma_1(\mu), \tag{2.27}$$

$$\hat{C}^o(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \hat{C}^o(\mu,\eta)\, h(\mu,\eta) d\eta}{\int_{\eta_L}^{\eta_H} h(\mu,\eta) d\eta} = \left[1-\Phi\left(\mu, \frac{1+\hat{r}}{\mu}\right)\right]\left[\frac{1+\hat{r}}{\mu}+\lambda\right]\hat{w}\Gamma_1(\mu), \tag{2.28}$$

$$\hat{A}^p(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \hat{A}^p(\mu,\eta)\, h(\mu,\eta) d\eta}{\int_{\eta_L}^{\eta_H} h(\mu,\eta) d\eta} = \left[1-\Phi\left(\mu, \frac{1+\hat{r}}{\mu}\right)\left[1+\frac{\lambda\mu}{1+\hat{r}}\right]\right]\hat{w}\Gamma_1(\mu), \tag{2.29}$$

$$\mathbb{E}\Lambda(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \mathbb{E}\Lambda(\mu,\eta)\, h(\mu,\eta) d\eta}{\int_{\eta_L}^{\eta_H} h(\mu,\eta) d\eta}. \tag{2.30}$$

In panel (a) we find that $\hat{C}^y(\mu)$ is increasing in μ. This result is the opposite of the findings reported by Heijdra and Reijnders (2012, p. 321) who assume that all individuals have the same labour productivity (i.e., $\sigma_\eta^2 = 0$ in their model). In our model, for a given productivity level η, youth consumption is decreasing in the survival probability (see Figure 2.2). But as a result of the positive correlation between η and μ, healthy agents also tend to be wealthy agents who consume more in youth as a result. Referring to equation (2.27), the term $\Phi\left(\mu, \frac{1+\hat{r}}{\mu}\right)\left[1+\frac{\lambda\mu}{1+\hat{r}}\right]$ is decreasing in μ but the $\Gamma_1(\mu)$ term is increasing in μ (see Lemma 1(vi))). Due to the strong correlation between μ and η the latter effect dominates the former, thus ensuring that $\hat{C}^y(\mu)$ is increasing in the survival probability.

As panel (b) shows, the profile for old-age consumption $\hat{C}^o(\mu)$ is also increasing in μ. Again this result is reversed if all agents feature the same labour productivity, as can be easily verified with the aid of Figure 2.2. In panel (c) we find that $\hat{A}^p(\mu)$ is increasing in μ. This result even holds if $\sigma_\eta^2 = 0$ (so that $\Gamma_1(\mu)$ is a constant) because $1-\Phi\left(\mu, \frac{1+\hat{r}}{\mu}\right)\left[1+\frac{\lambda\mu}{1+\hat{r}}\right]$ is increasing in μ. Finally, as panel (d) illustrates, $\mathbb{E}\hat{\Lambda}(\mu)$ is increasing in the survival probability. Intuitively, for a given productivity level η individual lifetime utility is increasing in μ (people like surviving into old-age). Furthermore, μ and η are positively correlated thus strengthening the positive link between utility and health.

2.3 Informational asymmetry in the private annuity market

In the previous section we have studied the steady state of an economy populated by heterogeneous individuals facing longevity risk and differing in terms of their innate labour productivity. With full information about the health status of individuals, annuity firms can effectively segment the market for private annuities and offer these insurance products at a price that is actuarially fair for all individuals. In this section we study the less pristine—and arguably much more realistic—scenario under which information regarding a person's health is not perfectly observable by insurance firms. Indeed, from here on we drop Assumption (A0) and replace it by the following alternative assumptions:

(A3) Health status and productivity are private information of the annuitant. The distribution of health and productivity types in the population, $H(\mu, \eta)$, is common knowledge.

(A4) Annuitants can buy multiple annuities for different amounts and from different annuity firms. Individual annuity firms cannot monitor their clients' wage income or annuity holdings with other firms.

As is explained by Heijdra and Reijnders (2012, pp. 325–6), in this *Asymmetric Information* case (abbreviated as AI) the market for private annuities is characterized by a pooling equilibrium. In this equilibrium there is a single pooled annuity rate, $\bar{r}^p_{t+1}$, which applies to all purchasers of private annuities. Lacking information about an individual's health and productivity, the annuity company cannot obtain full information revelation by setting both price and quantity. As a result, Pauly's (1974) linear pricing concept is the relevant one.[4] A second feature of the pooling equilibrium is that there typically are unhealthy agents who drop out of the annuity market altogether and face binding borrowing constraints. Indeed, since an individual's human wealth is proportional to his/her labour productivity, and individual consumption is decreasing in the survival rate, there may exist a cut-off survival probability, μ^{bc}_t, below which individuals would like to go short on annuities. But this is impossible because in doing so they would reveal their poor health status and obtain an offer they cannot possibly accept from annuity firms (more on this below).[5]

The pooled annuity rate, $\bar{r}^p_{t+1}$, is determined as follows. We assume that the cut-off health type is μ^{bc}_t such that consumers with health type $\mu_L \leq \mu < \mu^{bc}_t$ purchase no annuities. Net savers feature a survival probability such that $\mu^{bc}_t \leq \mu \leq \mu_H$ and purchase annuities. The zero-profit condition for the private annuity market is given by:

$$(1+r_{t+1}) \int_{\eta_L}^{\eta_H} \int_{\mu^{bc}_t}^{\mu_H} L_t(\mu,\eta) A^p_t(\mu,\eta) d\mu d\eta = (1+\bar{r}^p_{t+1}) \int_{\eta_L}^{\eta_H} \int_{\mu^{bc}_t}^{\mu_H} \mu L_t(\mu,\eta) A^p_t(\mu,\eta) d\mu d\eta, \tag{2.31}$$

where $1 + r_{t+1}$ is the gross rate of return on physical capital, $1 + \bar{r}^p_{t+1}$ is the gross rate of return on private annuities, $L_t(\mu, \eta)$ is the density of type (μ, η) consumers

[4]See also Abel (1986), Walliser (2000), Palmon and Spivak (2007), and Sheshinski (2008) on linear pricing of annuities.

[5]Villeneuve formulates a partial equilibrium model with heterogeneous survival rates (and identical labour productivity). He argues that only one insurance market can be active at any time, i.e. either the annuity market or the life-insurance market is active but not both. If there is no demand for life insurance in the full information case—as is the case in our model of the closed economy—then adverse selection in the market for private annuities cannot result in the activation of the life insurance market (2003, p. 534).

in period t, and $A_t^p(\mu,\eta)$ is the density of private annuities that is purchased by such agents. The gross returns from the annuity savings of all annuitants in period t (left-hand side of (2.31)) are redistributed to the surviving annuitants in the form of insurance claims in period $t+1$ (right-hand side of (2.31)). It follows that the pooling rate equals:

$$1+\bar{r}_{t+1}^{p}=\frac{1+r_{t+1}}{\bar{\mu}_t^p}, \tag{2.32}$$

where $\bar{\mu}_t^p$ denotes the asset-weighted average survival rate of annuity purchasers:

$$\bar{\mu}_t^p \equiv \int_{\mu_t^{bc}}^{\mu_H} \mu\omega_t(\mu)d\mu, \quad \omega_t(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} A_t^p(\mu,\eta)h(\mu,\eta)d\eta}{\int_{\eta_L}^{\eta_H}\int_{\mu_t^{bc}}^{\mu_H} A_t^p(\mu,\eta)h(\mu,\eta)d\mu d\eta}. \tag{2.33}$$

In view of the fact that the asset-weighted survival rate is such that $\mu_t^{bc} < \bar{\mu}_t^p < \mu_H < 1$, it follows from (2.32) that $\bar{r}_{t+1}^p$ exceeds r_{t+1} so that all net savers will completely annuitize their wealth. Hence, Yaari's (1965) classic result also holds in the pooled annuity market.

The pooling rate (2.32) is demographically unfair because it is based on the *asset-weighted* survival rate $\bar{\mu}_t^p$ rather than on the *average* survival rate in the population $\bar{\mu}$. The demographically fair pooling rate is given by:

$$1+\bar{r}_{t+1}^{df}=\frac{1+r_{t+1}}{\bar{\mu}}, \tag{2.34}$$

and, since $\bar{\mu} < \bar{\mu}_t^p$ (see Appendix B), it follows readily from the comparison of (2.32) and (2.34) that $\bar{r}_{t+1}^p < \bar{r}_{t+1}^{df}$. In our numerical exercise we follow Walliser (2000, p. 380) by constructing an adverse selection index AS_t (or 'load factor') which shows by how much the asking price of an annuity insurance company exceeds the demographically fair price:

$$AS_t \equiv \frac{1/(1+\bar{r}_{t+1}^p)}{1/(1+\bar{r}_{t+1}^{df})} = \frac{\bar{\mu}_t^p}{\bar{\mu}}. \tag{2.35}$$

As a result of adverse selection in the private annuity market, AS_t exceeds unity. Furthermore, the larger is AS_t, the more severe is the adverse selection problem.

Under the maintained assumption that $\mu_L < \mu_t^{bc} < \mu_H$, there are two types of agents in the economy. Individuals with a relatively low survival probability ($\mu_L \le \mu < \mu_t^{bc}$) will face a binding borrowing constraint, whilst healthier individuals ($\mu_t^{bc} \le \mu \le \mu_H$) will be net savers. It follows that constrained individuals simply consume their endowment incomes in the two periods:

$$C_t^y(\mu,\eta) = w_t(\eta), \tag{2.36}$$

$$C_{t+1}^o(\mu,\eta) = \lambda w_{t+1}(\eta). \tag{2.37}$$

For unconstrained individuals the consolidated budget constraint in a pooled annuity market is given by:

$$C_t^y(\mu,\eta)+\frac{C_{t+1}^o(\mu,\eta)}{1+\bar{r}_{t+1}^p} = w_t(\eta)+\frac{\lambda w_{t+1}(\eta)}{1+\bar{r}_{t+1}^p}, \tag{2.38}$$

where $\bar{r}^p_{t+1}$ is the pooling rate of interest. Such consumers choose $C^y_t(\mu,\eta)$ and $C^o_{t+1}(\mu,\eta)$ in order to maximize expected lifetime utility (2.2) subject to the budget constraint (2.38). The optimal consumption plans and annuity demand are fully characterized by:

$$C^y_t(\mu,\eta) = \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}^p_t}\right)\left[w_t(\eta) + \frac{\lambda\bar{\mu}^p_t w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.39}$$

$$\frac{\bar{\mu}^p_t C^o_{t+1}(\mu,\eta)}{1+r_{t+1}} = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}^p_t}\right)\right]\left[w_t(\eta) + \frac{\lambda\bar{\mu}^p_t w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.40}$$

$$A^p_t(\mu,\eta) = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}^p_t}\right)\right] w_t(\eta) - \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}^p_t}\right)\frac{\lambda\bar{\mu}^p_t w_{t+1}(\eta)}{1+r_{t+1}}, \tag{2.41}$$

where we have used the expression for the pooled annuity rate as given in (2.32).

The optimal consumption choices of different types of consumers are illustrated in Figure 2.4. Just as for the FI case we only illustrate the choices made by the four extreme types, unhealthy and healthy lowest-skilled (μ_L,η_L) and (μ_H,η_L), and unhealthy and healthy highest-skilled (μ_L,η_H) and (μ_H,η_H). In view of (2.38) the location of an individual's lifetime budget constraint only depends on the person's productivity level, so that LBC(η_L) and LBC(η_H) are parallel. As before the income endowment line is given by IEL, so that the two relevant endowment points are given by, respectively, points E_L and E_H. The consumption Euler equation for unconstrained consumers operating in a pooled annuity market is given by:

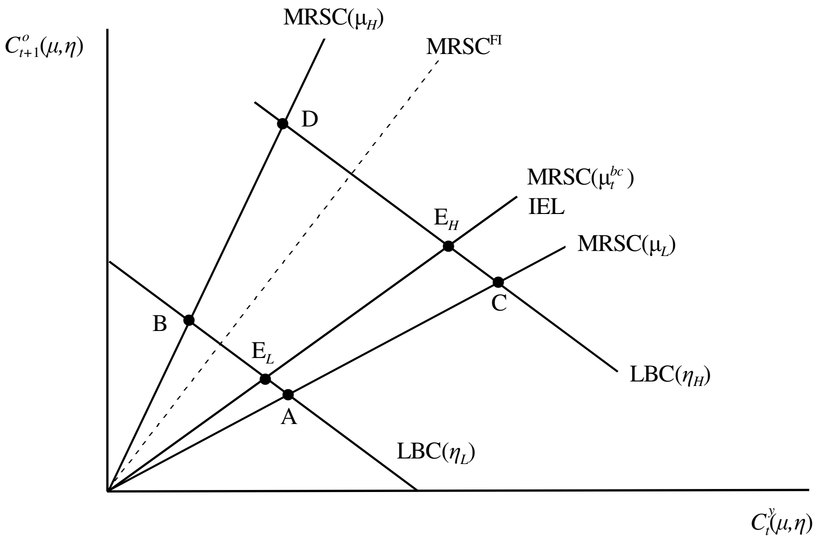
$$\frac{U'(C^y_t(\mu,\eta))}{\beta U'(C^o_{t+1}(\mu,\eta))} = \mu\left(1+\bar{r}^p_{t+1}\right) = \frac{\mu}{\bar{\mu}^p_t}(1+r_{t+1}), \tag{2.42}$$

where we have used (2.32) to get from the first to the second equality. Using the CRRA felicity function stated in (2.3), we easily find that the Euler equation is a straight line from the origin with a slope that depends positively on μ. In Figure 2.4 we have drawn the Euler equations as MRSC(μ_H) and MRSC(μ_L). Since MRSC(μ_H) lies to the left of IEL, points B and D denote the optimal (unconstrained) consumption points for, respectively, the lowest-skilled and highest-skilled consumers. In contrast, since MRSC(μ_L) lies to the right of IEL, points A and C are infeasible as they would involve going short on annuities. It follows that all lowest-health individuals face borrowing constraints. Furthermore, the Euler equation (2.42) that coincides with the IEL, MRSC(μ^{bc}_t), determines the cut-off health type μ^{bc}_t:

$$\mu^{bc}_t = \frac{\bar{\mu}^p_t U'(w_t(\eta))}{(1+r_{t+1})\beta U'(\lambda w_{t+1}(\eta))}. \tag{2.43}$$

Unconstrained consumers are located in the area E_LBDE_H whilst constrained individuals are bunched on the line segment $\mathrm{E}_L\mathrm{E}_H$. It is worth noting that μ^{bc}_t depends on the current and future capital intensity in the economy via factor prices. Given the specification of preferences and technology, however, μ^{bc}_t does not depend on η itself.

Figure 2.4: Consumption-saving choices under asymmetric information



Legend LBC(η_j) is the lifetime budget constraint for an individual with productivity η_j. IEL is the income endowment line and agents are located on the line segment $\mathrm{E}_L\mathrm{E}_H$. MRSC(μ_i) is the consumption Euler equation for an individual with survival rate μ_i facing a pooled annuity rate of interest $\bar{r}^p_{t+1}$. For individuals with $\mu^{bc}_t \leq \mu \leq \mu_H$ optimal consumption is located at the intersection of MRSC(μ_i) and LBC(η_j). All other individuals face borrowing constraints and consume along $\mathrm{E}_L\mathrm{E}_H$.

In the presence of binding borrowing constraints, the capital accumulation identity (2.23) is augmented to:

$$K_{t+1} = L_t \int_{\eta_L}^{\eta_H} \int_{\mu_t^{bc}}^{\mu_H} A_t^p(\mu,\eta) h(\mu,\eta) d\eta d\mu. \tag{2.44}$$

By substituting the demand for annuities (2.41) into (2.44) we obtain the fundamental difference equation for the capital intensity:

$$k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\left[\int_{\mu_t^{bc}}^{\mu_H}\left[w_t - \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\left[w_t + \frac{\lambda\bar{\mu}_t^p w_{t+1}}{1+r_{t+1}}\right]\right] h_\mu(\mu)\Gamma_1(\mu) d\mu\right], \tag{2.45}$$

where $\Gamma_1(\mu)$ is the conditional mean of η (defined in Lemma 1(vi) above), and the factor prices follow from (2.20)–(2.21).

The main features of the steady-state AI equilibrium are reported in column (b) of Table 2.2. As a result of asymmetric information in the annuity market, output per efficiency unit of labour drops by 1.60% ($\hat{y} = 9.840$) whilst the steady-state capital intensity falls by 5.71% ($\hat{k} = 0.373$). The decrease in the capital intensity causes the annual interest rate to rise by 12 basis points ($\hat{r}^a = 5.11\%$) and the wage rate to fall by 1.60%. So despite the fact that only 5.83% of young individuals face binding borrowing constraints (see $\widehat{BC}$), the macroeconomic effects of information asymmetry are far from trivial in size. The adverse selection index, as defined in (2.35) above, equals $\widehat{AS} = 1.31$ and the asset-weighted average survival rate of annuitants equals $\hat{\bar{\mu}}^p = 0.66$. Finally, as the welfare indicators at the bottom of Table 2.2 reveal, under asymmetric information unhealthy individuals are worse off while their healthy cohort members are better off than under the FI case. The information asymmetry redistributes resources from unhealthy to healthy agents.

In Figure 2.3 we depict with dashed lines the steady-state profiles for youth consumption, old-age consumption, annuity demand, and expected utility. Just as for the FI case these profiles have been averaged over η:

$$\hat{C}^y(\mu) = \left[1 - \mathbb{I}_{AI}(\mu) + \mathbb{I}_{AI}(\mu)\Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\left[1 + \frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right]\right]\hat{w}\Gamma_1(\mu), \tag{2.46}$$

$$\hat{C}^o(\mu) = \left[[1 - \mathbb{I}_{AI}(\mu)]\lambda + \mathbb{I}_{AI}(\mu)\left[1 - \Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\right]\left[\frac{1+\hat{r}}{\hat{\bar{\mu}}^p} + \lambda\right]\right]\hat{w}\Gamma_1(\mu), \tag{2.47}$$

$$\hat{A}^p(\mu) = \mathbb{I}_{AI}(\mu)\left[1 - \Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\left[1 + \frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right]\right]\hat{w}\Gamma_1(\mu), \tag{2.48}$$

where $\mathbb{I}_{AI}(\mu) = 0$ for $\mu_L \leq \mu < \hat{\mu}^{bc}$ and $\mathbb{I}_{AI}(\mu) = 1$ for $\hat{\mu}^{bc} \leq \mu \leq \mu_H$. In panel (a) we find that youth consumption $\hat{C}^y(\mu)$ is increasing in μ. Interestingly, for μ close to $\hat{\mu}^{bc}$ youth consumption is higher under AI than for the FI case. Young individuals facing borrowing constraints are unable to smooth consumption in the AI case and just consume their endowment income. Net savers featuring a survival probability close to $\hat{\mu}^{bc}$ purchase virtually no annuities at all as the pooling rate is unattractive to them—see panel (c). For higher levels of μ annuity demands are higher and saving for old-age increases. In panel (b) we show that the healthiest agents consume more during old-age under AI compared to FI. In panel (d) we find that the healthiest individuals are actually better off under AI than under FI. The information asymmetry benefits such individuals.

2.4 Public annuities to the rescue?

In the adverse selection economy studied in the previous section relatively unhealthy annuitants face a disadvantageous pooling rate of interest on their annuities. In essence such individuals are subsidizing their healthy cohort members through the annuity market. Following Abel (1987) we now extend our model by introducing a fully-funded mandatory social security system that is run by the government.[6] Such a system is immune to adverse selection because all individuals are forced to participate in it—the government possesses the power to tax. In particular, every individual pays a social security tax θ (such that $0 < \theta < 1$) and receives a retirement pension upon surviving into old-age. We assume that the pension contribution is proportional to wage income. Like the private sector, the government cannot observe an individual's health status though it can measure a person's income. It follows that the pension contribution can be written as $A_t^s(\eta) = \theta w_t(\eta)$. Total pension contributions amount to $A_t^s = \theta\bar{\eta}w_t L_t$ and are invested in the capital market earning a gross rate of return equal to $1 + r_{t+1}$. In the next period the returns $R_{t+1} = (1 + r_{t+1})A_t^s$ are paid out to surviving agents. Under this funded pension system redistribution takes place between agents of the same birth cohort (from those who die to survivors). Hence, social security plays the role of public annuities. In this section we consider three prototypical types of pension systems. The difference lies in the method in which the returns are distributed to surviving individuals.

- Pension system A: pension receipts during old-age are proportional to contributions made during youth.
- Pension system B: pension contributions of η types are distributed during old-age to surviving η types.
- Pension system C: pension receipts are the same in absolute value for all surviving agents.

2.4.1 Pension system A

Under system A pension receipts are given by:

$$R_{t+1}^s(\eta) = \zeta\theta w_t(\eta), \tag{2.49}$$

[6]There is one important difference in that Abel (1987) restricts attention to the full information (FI) case in which perfect private annuities are available. In order not to unduly interrupt the flow of the paper, we present the FI results for our model in Appendix C.

where ζ is a parameter to be determined below. The clearing condition for the public annuity system is given in this case by:

$$(1 + r_{t+1})A_t^s = \int_{\eta_L}^{\eta_H} \int_{\mu_L}^{\mu_H} \mu R_{t+1}^s(\eta) L_t(\mu, \eta) d\mu d\eta. \tag{2.50}$$

The left-hand side of this expression is the total amount to be distributed to survivors and the right-hand side represents total pension payments. By substituting (2.49) into (2.50) and noting that $w_t(\eta) = \eta w_t$ and $L_t(\mu, \eta) = L_t h(\mu, \eta)$ we find the balanced-budget solution for ζ:

$$\zeta = \zeta_A \, \frac{1 + r_{t+1}}{\bar{\mu}}, \qquad \zeta_A \equiv \frac{\bar{\eta}\bar{\mu}}{\operatorname{cov}(\eta, \mu) + \bar{\eta}\bar{\mu}}, \tag{2.51}$$

where $\bar{\mu}$ is the average survival rate of the population and ζ_A is a constant (featuring $0 < \zeta_A < 1$ because $\operatorname{cov}(\eta, \mu)$ is positive). It follows from (2.51) that under pension system A the rate of return on social annuities falls short of the demographically fair social annuity yield, $(1+r_{t+1})/\bar{\mu}$, because health and productivity are positively correlated. Intuitively, the high contributors (featuring a high η) tend to live longer than average.

Just as in the adverse selection economy studied in the previous section individuals can buy private annuities in the pooled annuity market but some agents will face borrowing constraint. Constrained individuals simply consume their endowment incomes in the two periods:

$$C_t^y(\mu, \eta) = (1 - \theta) w_t(\eta), \tag{2.52}$$

$$C_{t+1}^o(\mu, \eta) = \lambda w_{t+1}(\eta) + R_{t+1}^s(\eta). \tag{2.53}$$

For unconstrained individuals the consolidated budget constraint in the presence of a pooled annuity market is given by:

$$C_t^y(\mu, \eta) + \frac{C_{t+1}^o(\mu, \eta)}{1 + \bar{r}_{t+1}^p} = (1 - \theta) w_t(\eta) + \frac{\lambda w_{t+1}(\eta) + R_{t+1}^s(\eta)}{1 + \bar{r}_{t+1}^p}, \tag{2.54}$$

where $\bar{r}_{t+1}^p$ is the pooling rate of interest. The pension system reduces current wage income but increases future income. Consumers choose $C_t^y(\mu, \eta)$ and $C_{t+1}^o(\mu, \eta)$ in order to maximize expected lifetime utility (2.2) subject to the budget constraint (2.54). The optimal consumption plans and annuity demands are fully characterized by:

$$C_t^y(\mu,\eta) = \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\left[(1-\theta)w_t(\eta) + \theta\zeta_A w_t(\eta)\frac{\bar{\mu}_t^p}{\bar{\mu}} + \frac{\lambda\bar{\mu}_t^p w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.55}$$

$$\frac{\bar{\mu}_t^p C_{t+1}^o(\mu,\eta)}{1+r_{t+1}} = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\right]\left[(1-\theta)w_t(\eta) + \theta\zeta_A w_t(\eta)\frac{\bar{\mu}_t^p}{\bar{\mu}} + \frac{\lambda\bar{\mu}_t^p w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.56}$$

$$A_t^p(\mu,\eta) = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\right](1-\theta)w_t(\eta) - \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\left[\theta\zeta_A w_t(\eta)\frac{\bar{\mu}_t^p}{\bar{\mu}} + \frac{\lambda\bar{\mu}_t^p w_{t+1}(\eta)}{1+r_{t+1}}\right], \tag{2.57}$$

where we have used the expression for the pooled annuity rate as given in (2.32). The social annuity system affects an individual's human wealth at birth (the term in square brackets on the right-hand side of (2.55)) but it is not a priori clear in which direction. Indeed, the *effective* pension contribution rate is:

$$\theta_t^n \equiv \theta\left(1 - \zeta_A\frac{\bar{\mu}_t^p}{\bar{\mu}}\right). \tag{2.58}$$

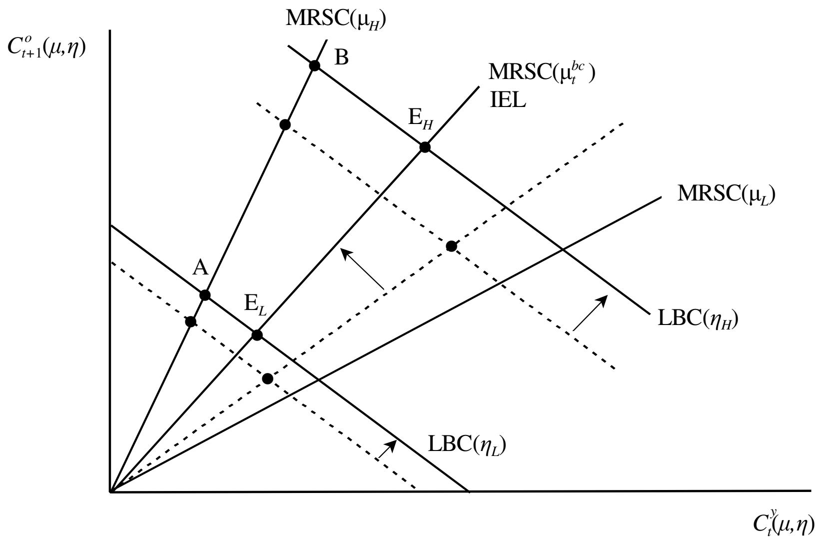
On the one hand, with a positive correlation between health and ability ζ_A is such that $0 < \zeta_A < 1$. On the other hand, the survival rate of private annuitants exceeds the population-wide average survival rate, i.e. $\bar{\mu}_t^p/\bar{\mu} > 1$. It thus follows that θ_t^n is ambiguous in sign. In this paper we focus on the case for which θ_t^n is negative so that, ceteris paribus factor prices and the pooled survival rate, human wealth is increased as a result of the public pension system.[7]

The optimal consumption choices can be explained with the aid of Figure 2.5. To facilitate the comparison with the AI case we keep factor prices and the pooled survival rate at the levels for that case. Hence the diagram shows the partial equilibrium effects on individual choices of the introduction of a pension system. The dashed lines correspond to the AI case. As a result of the public pension system the lifetime budget constraints shift outward (because $\theta_t^n < 0$), more so the higher is η. The income endowment line rotates in a counter-clockwise fashion. Unconstrained individuals increase consumption during youth and old-age. In contrast, constrained individuals are forced to consume less during youth. Such agents are bunched along the line segment $\mathrm{E}_L\mathrm{E}_H$. Just as for the AI case there is a single cut-off value for the survival probability below which agents are facing borrowing constraints:

$$\mu_t^{bc} = \frac{\bar{\mu}_t^p U'((1-\theta)w_t(\eta))}{(1+r_{t+1})\beta U'\left(\lambda w_{t+1}(\eta) + \theta\zeta_A\ \frac{1+r_{t+1}}{\bar{\mu}} w_t(\eta)\right)}. \tag{2.59}$$

[7] In the numerical simulations $\zeta_A = 0.9569$ and $\bar{\mu} = 0.5$. Hence the effective pension contribution is negative for any $\bar{\mu}_t^p$ exceeding $\bar{\mu}/\zeta_A = 0.5225$. This condition is easily satisfied. See also Figure 2.9(c) for an illustration of effective contribution rates under the different pension systems.

Figure 2.5: Consumption-saving choices under pension system A



Legend LBC(η_j) is the lifetime budget constraint for an individual with productivity η_j. IEL is the income endowment line and agents are located on the line segment $\mathrm{E}_L\mathrm{E}_H$. MRSC(μ_i) is the consumption Euler equation for an individual with survival rate μ_i facing a pooled annuity rate of interest $\bar{r}^p_{t+1}$. For individuals with $\mu^{bc}_t \leq \mu \leq \mu_H$ optimal consumption is located at the intersection of MRSC(μ_i) and LBC(η_j). All other individuals face borrowing constraints and consume along $\mathrm{E}_L\mathrm{E}_H$. The dashed lines visualize the corresponding schedules for the AI case. Factor prices are held the same for SA and AI to facilitate the comparison.

Because wages and pension receipts are proportional to η and the felicity function is homothetic, it follows from (2.59) that μ^{bc}_t does not depend on η. As is clear from the diagram, the introduction of public pensions will increase the population fraction of people facing borrowing constraints.

In order to glean the general equilibrium effects of introducing a public pension system we must formulate the capital accumulation identity. Since public and private annuities are invested in the capital markets, the accumulation equation takes the following format:

$$K_{t+1} = L_t \left[A^s_t + \int_{\eta_L}^{\eta_H} \int_{\mu^{bc}_t}^{\mu_H} A^p_t(\mu,\eta) h(\mu,\eta) d\eta d\mu \right]. \tag{2.60}$$

By substituting the demand for annuities (2.57) into (2.60) we obtain the fundamental difference equation for the capital intensity:

$$k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\left[\theta\bar{\eta}w_t + \int_{\mu_t^{bc}}^{\mu_H}\left((1-\theta)w_t - \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\right.\right.$$
$$\left.\left.\cdot\left[(1-\theta)w_t + \theta\zeta_A w_t\frac{\bar{\mu}_t^p}{\bar{\mu}} + \frac{\lambda\bar{\mu}_t^p w_{t+1}}{1+r_{t+1}}\right]\right)h_\mu(\mu)\Gamma_1(\mu)d\mu\right], \tag{2.61}$$

where $\Gamma_1(\mu)$ is the conditional mean of η (defined in Lemma 1(vi) above), and the factor prices follow from (2.20)–(2.21).

The main features of the steady-state equilibrium with pension system A (labeled SA) are reported in columns (c)-(d) of Table 2.2. In column (c) the contribution rate equals $\theta = 0.010$ which means that the system is relatively small as the income replacement rate during retirement, $\xi_{SA} \equiv \theta\zeta_A(1+\hat{r})/[(1-\lambda)\bar{\mu}]$, is only about 0.3812. In column (d) the contribution rate equals $\theta = 0.025$ which results in a large pension system, i.e. $\xi_{SA} = 0.9776$. Comparing columns (b) and (d) we find that output per efficiency unit of labour drops by 1.62% ($\hat{y} = 9.680$) whilst the steady-state capital intensity falls by 5.76% ($\hat{k} = 0.351$). As a result of the decrease in the capital intensity, the annual interest rate rises by 11 basis points ($\hat{r}^a = 5.22\%$) whilst the wage rate falls by 1.6%. The proportion of constrained individual rises from 5.83% to 17.66%. The adverse selection index, as defined in (2.35) above, increases to $\widehat{AS} = 1.39$ and the asset-weighted average survival rate of annuitants rises to $\hat{\bar{\mu}}^p = 0.70$. Despite the fact that the rate of return on capital increases, the return on private annuities decreases slightly because the pooled survival rate $\hat{\bar{\mu}}^p$ increases by more. Finally, as the welfare indicators at the bottom of Table 2.2 reveal, under pension system A all individuals are worse off compared to the AI case. The pension system crowds out capital and exacerbates the adverse selection problem in the market for private annuities.

In Figure 2.6 we use solid lines to depict the profiles for youth and old-age consumption, annuity demand, and utility (averaged over η) for the SA case. These are given by:

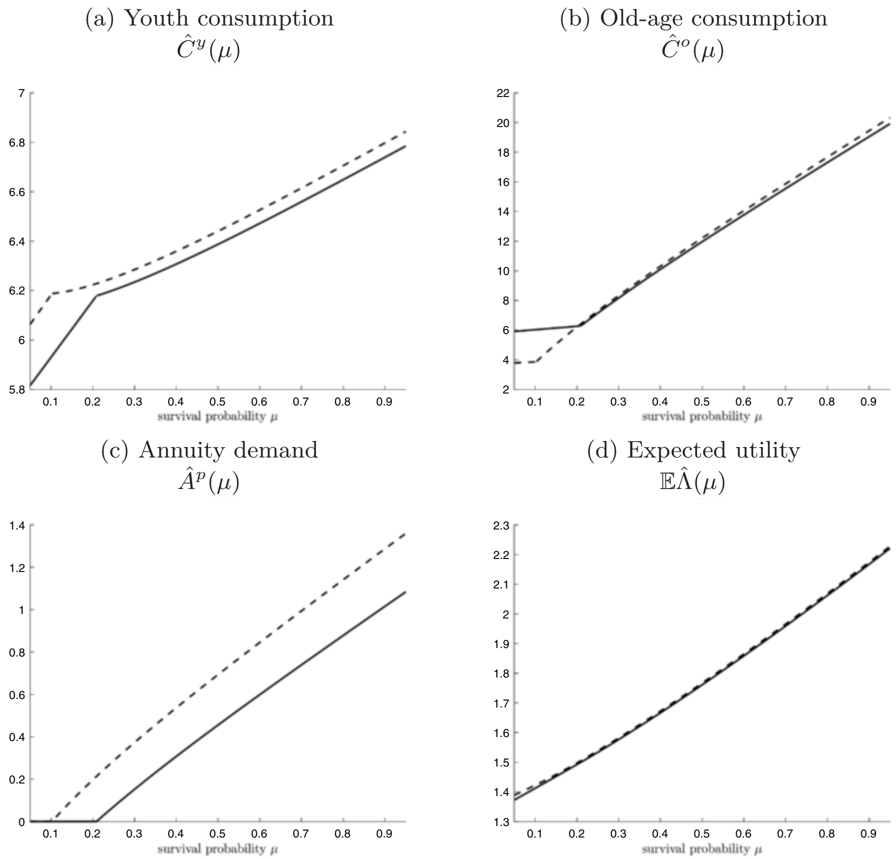
$$\frac{\hat{C}^y(\mu)}{\hat{w}} = \left[1 - \mathbb{I}_{SA}(\mu) + \mathbb{I}_{SA}(\mu)\Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\left[1-\theta+\theta\zeta_A\frac{\hat{\bar{\mu}}^p}{\bar{\mu}} + \frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right]\right]\Gamma_1(\mu), \tag{2.62}$$

$$\frac{\hat{C}^o(\mu)}{\hat{w}} = [1-\mathbb{I}_{SA}(\mu)]\left[\lambda + \theta\zeta_A\frac{1+\hat{r}}{\bar{\mu}}\right]\Gamma_1(\mu) + \mathbb{I}_{SA}(\mu)\left[1 - \Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\right]$$
$$\cdot\left[\left(1-\theta+\theta\zeta_A\frac{\hat{\bar{\mu}}^p}{\bar{\mu}}\right)\frac{1+\hat{r}}{\hat{\bar{\mu}}^p} + \lambda\right]\Gamma_1(\mu), \tag{2.63}$$

$$\frac{\hat{A}^p(\mu)}{\hat{w}} = \mathbb{I}_{SA}(\mu)\left[1-\theta-\Phi\left(\mu, \frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\left[1-\theta+\theta\zeta_A\frac{\hat{\bar{\mu}}^p}{\bar{\mu}} + \frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right]\right]\Gamma_1(\mu), \tag{2.64}$$

where $\mathbb{I}_{SA}(\mu) = 0$ for $\mu_L \leq \mu < \hat{\mu}^{bc}$ and $\mathbb{I}_{AI}(\mu) = 1$ for $\hat{\mu}^{bc} \leq \mu \leq \mu_H$. The dashed lines in Figure 2.6 correspond to the profiles for the AI case. Youth consumption,

Figure 2.6: Steady-state profiles under pension system A



Legend The solid lines depict the steady-state profiles under pension system A (SA), and the dashed lines visualize the profiles for the asymmetric information (AI) case without pensions. In both cases adverse selection results in a single pooling rate of interest on annuities, $\bar{r}^p_{t+1}$, and agents with poor health face binding borrowing constraints. The SA case has been drawn for a large system featuring $\theta = 0.025$.

annuity demand, and lifetime utility are all lower under SA than under AI. Old-age consumption is higher under AI for most borrowing constrained individuals.

2.4.2 Pension system B

Under pension system B the government uses information on a person's wage income to deduce that individual's innate ability. It uses its knowledge of η by setting pension receipts according to the following rule:

$$R^s_{t+1}(\eta) = \zeta(\eta)\theta w_t(\eta), \tag{2.65}$$

where $\zeta(\eta)$ is a function to be determined below. For each ability level η, the budget constraint for the public pension system is given by:

$$(1 + r_{t+1})\theta w_t(\eta) \int_{\mu_L}^{\mu_H} L_t(\mu, \eta) d\mu = \int_{\mu_L}^{\mu_H} \mu R^s_{t+1}(\eta) L_t(\mu, \eta) d\mu. \tag{2.66}$$

The left-hand side of this expression is the total amount to be distributed to type η survivors whilst the right-hand side represents total pension payments to such individuals. Under this system public annuities are such that longevity risk is shared among individuals of the same productivity type. By substituting (2.65) into (2.66) and noting that $w_t(\eta) = \eta w_t$ and $L_t(\mu, \eta) = L_t h(\mu, \eta)$ we find the balanced-budget solution for $\zeta(\eta)$:

$$\zeta(\eta) = \zeta_B(\eta)\, \frac{1 + r_{t+1}}{\bar{\mu}}, \qquad \zeta_B(\eta) \equiv \frac{\bar{\mu}}{\bar{\mu} + \xi\sigma^2_\mu(\eta - \bar{\eta})}. \tag{2.67}$$

For relatively productive individuals (featuring $\eta > \bar{\eta}$) the rate of return on social annuities falls short of the actuarially fair social annuity yield, $(1 + r_{t+1})/\bar{\mu}$, because such people tend to have a relatively high survival rate. In contrast, for relatively unproductive individuals (with $\eta < \bar{\eta}$) the rate of return on social annuities is better than the actuarially fair social annuity yield because such people tend to have a relatively low survival rate.

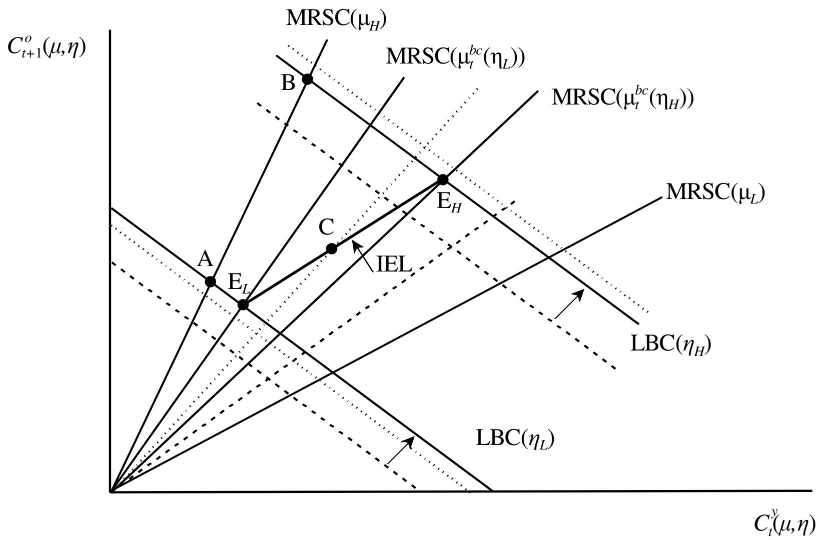
Individuals facing a binding borrowing constraint consume according to (2.52)–(2.53) with $R^s_{t+1}(\eta)$ as stated in (2.65) and (2.67). For unconstrained individuals the optimal consumption plans and annuity demands are fully characterized by:

$$C^y_t(\mu, \eta) = \Phi\left(\mu, \frac{1 + r_{t+1}}{\bar{\mu}^p_t}\right) \left[(1 - \theta)w_t + \theta\zeta_B(\eta)w_t \frac{\bar{\mu}^p_t}{\bar{\mu}} + \frac{\lambda\bar{\mu}^p_t w_{t+1}}{1 + r_{t+1}}\right] \eta, \tag{2.68}$$

$$\begin{aligned} \frac{\bar{\mu}^p_t C^o_{t+1}(\mu, \eta)}{1 + r_{t+1}} &= \left[1 - \Phi\left(\mu, \frac{1 + r_{t+1}}{\bar{\mu}^p_t}\right)\right] \left[(1 - \theta)w_t + \theta\zeta_B(\eta)w_t \frac{\bar{\mu}^p_t}{\bar{\mu}}\right. \\ &\qquad \left. + \frac{\lambda\bar{\mu}^p_t w_{t+1}}{1 + r_{t+1}}\right] \eta, \end{aligned} \tag{2.69}$$

$$\begin{aligned} A^p_t(\mu, \eta) &= \left[1 - \Phi\left(\mu, \frac{1 + r_{t+1}}{\bar{\mu}^p_t}\right)\right] (1 - \theta)\eta w_t \\ &\qquad - \Phi\left(\mu, \frac{1 + r_{t+1}}{\bar{\mu}^p_t}\right) \left[\theta\zeta_B(\eta)w_t \frac{\bar{\mu}^p_t}{\bar{\mu}} + \frac{\lambda\bar{\mu}^p_t w_{t+1}}{1 + r_{t+1}}\right] \eta, \end{aligned} \tag{2.70}$$

Figure 2.7: Consumption-saving choices under pension system B



Legend LBC(η_j) is the lifetime budget constraint for an individual with productivity η_j. IEL is the income endowment line and agents are located on the line segment $\mathrm{E}_L\mathrm{E}_H$. MRSC(μ_i) is the consumption Euler equation for an individual with survival rate μ_i facing a pooled annuity rate of interest $\bar{r}^p_{t+1}$. The dashed and dotted lines visualize the corresponding schedules for the AI and SA cases respectively. Factor prices are held the same for SB and AI to facilitate the comparison. An individual with productivity η_j faces borrowing constraints if $\mu < \mu^{bc}_t(\eta_j)$ and is unconstrained otherwise.

where we have substituted $w_t(\eta) = \eta w_t$ and used the expression for the pooled annuity rate as given in (2.32).

The optimal consumption choices can be explained with the aid of Figure 2.7. Just as before we focus on the four extreme types. The solid lines depict the situation under pension system B. For purposes of reference the dashed lines in the diagram represent the AI case (without pensions) whilst the thin dotted lines represent the SA case. We keep factor prices constant at their AI levels. Under pension system B the IEL pivots around some point C on the old IEL line for the SA case. Intuitively this is because system B incorporates *explicit* redistribution from high-ability to low-ability individuals and, as a result of the positive correlation between ability and health, *implicit* redistribution from healthy to unhealthy individuals. With asymmetric information in the private annuity market the pooling equilibrium causes a redistribution of resources from unhealthy to healthy individuals, i.e. from people who tend to be poor to individuals who tend to be rich. Pension system A does nothing to redress this phenomenon. In contrast, under system B the highskilled get a lower return on social annuities than the low-skilled do, so there is some redistribution from healthy to unhealthy individuals via that channel.

As is marked in the diagram, lowest-ability types experience borrowing constraint for $\mu < \mu_t^{bc}(\eta_L)$ whilst highest-ability individuals experience such constraints for $\mu < \mu_t^{bc}(\eta_H)$, where $\mu_t^{bc}(\eta_H) < \mu_t^{bc}(\eta_L)$. Mathematically, an individual with productivity η experiences a binding borrowing constraint if his/her survival probability falls short of $\mu_t^{bc}(\eta)$:

$$\mu_t^{bc}(\eta) = \frac{\bar{\mu}_t^p U'((1-\theta)\eta w_t)}{(1+r_{t+1})\beta U'\left(\lambda\eta w_{t+1} + \theta\eta\zeta_B(\eta)\ \frac{1+r_{t+1}}{\bar{\mu}} w_t\right)} \tag{2.71}$$

Despite the fact that the felicity function is homothetic and wages are proportional to η, μ_t^{bc} depends on η because productivity features nonlinearly in $\zeta_B(\eta)$.

In Figure 2.8 we illustrate the relationship between ability η and the critical survival rate $\mu_t^{bc}(\eta)$. The thin solid line represents the AI case for which $\hat{\mu}^{bc} = 0.1028$ and 5.83% of agents are constrained. The dashed line depicts the situation for the SA case (with $\theta = 0.025$) for which $\hat{\mu}^{bc} = 0.2090$ and 17.66% of agents are constrained. Finally, the thick solid line in Figure 2.8 illustrates the SB case. As is predicted by the theory there is a downward sloping relationship between η and $\hat{\mu}^{bc}$. For the lowest-ability types the cut-off value equals 0.2590 whereas it is equal to 0.1902 for the highest-ability individuals. So by engaging in redistribution from high-ability to low-ability individuals the policy maker worsens the incidence of borrowing constraints to the latter types.

In Figure 2.9 we compare some features of pension systems A and B. Panel (a) depicts the fair-rates shares ζ_A (a constant) and $\zeta_B(\eta)$ (downward sloping because of redistribution). Panel (b) shows that pensions receipts are increasing in ability for both systems. In panel (c) we depict the effective pension contribution rate $\theta_t^n(\eta)$. Under system A this is a negative constant, but under system B the effective rate is increasing in ability:

$$\theta_t^n(\eta) \equiv \theta\left(1 - \zeta_B(\eta)\frac{\bar{\mu}_t^p}{\bar{\mu}}\right). \tag{2.72}$$

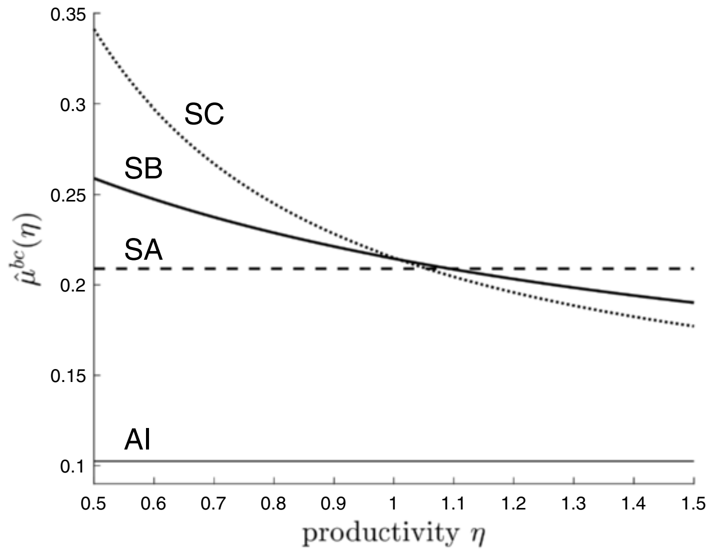
For our parameterization $\theta_t^n(\eta)$ remains negative for all ability levels, although barely so for the highest-ability types.

Under pension system B the capital accumulation identity is given by:

$$K_{t+1} = L_t\left[A_t^s + \int_{\eta_L}^{\eta_H}\int_{\mu_t^{bc}(\eta)}^{\mu_H} A_t^p(\mu,\eta)h(\mu,\eta)d\eta d\mu\right], \tag{2.73}$$

where $\mu_t^{bc}(\eta)$ is determined in (2.71) and is illustrated in Figure 2.8. By substituting the demand for annuities (2.70) into (2.73) we obtain the fundamental difference

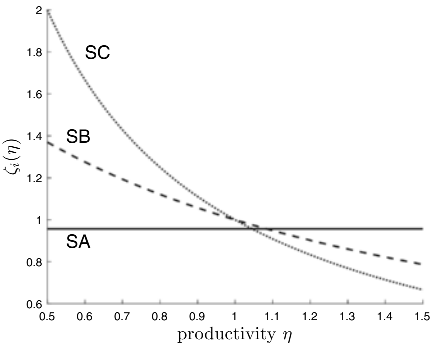
Figure 2.8: Ability and borrowing constraints



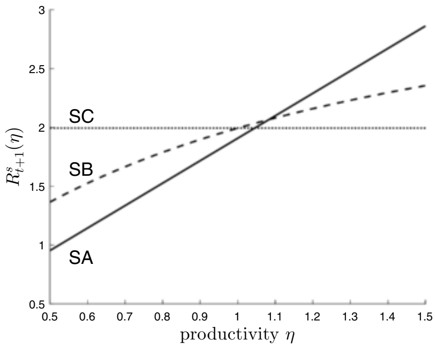
Legend Under pension systems B and C the critical level of the survival rate below which borrowing constraints become active, $\mu_t^{bc}(\eta)$, depends negatively on the individual's productivity η. In Figure 2.7 the income endowment points no longer lie along a ray from the origin.

Figure 2.9: Comparing pension systems

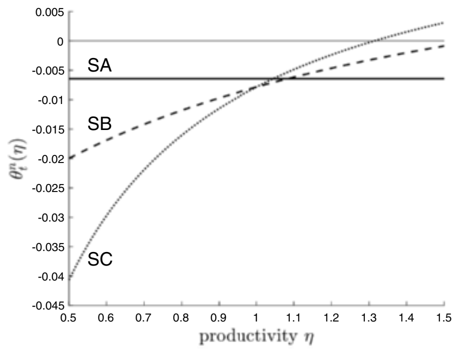
(a) Fair-rate share
$\zeta_i(\eta)$



(b) Pension receipt
$R^s_{t+1}(\eta)$



(c) Effective pension contribution rate
$\theta^n_t(\eta)$



Legend The fair-rate share $\zeta_i(\eta)$ measures the individual's gross yield on social annuities under pension system i expressed as a share of the actuarially fair yield, $(1+\bar{r}^p_{t+1})/\bar{\mu}$. A negative value for the effective pension contribution rate $\theta^n_t(\eta)$ implies that the pension system makes individuals wealthier in a partial equilibrium sense.

equation for the capital intensity:

$$k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\left[\theta\bar{\eta}w_t + \int_{\eta_L}^{\eta_H}\int_{\mu_t^{bc}(\eta)}^{\mu_H}\left((1-\theta)w_t - \Phi\left(\mu, \frac{1+r_{t+1}}{\bar{\mu}_t^p}\right)\right.\right.$$
$$\left.\left.\cdot\left[(1-\theta)w_t + \theta\zeta_B(\eta)w_t\frac{\bar{\mu}_t^p}{\bar{\mu}} + \frac{\lambda\bar{\mu}_t^p w_{t+1}}{1+r_{t+1}}\right]\right)\eta h(\mu,\eta)d\mu d\eta\right]. \tag{2.74}$$

The main features of the steady-state equilibrium with a small and large pension system B (labeled SB) are reported in, respectively, columns (e) and (f) of Table 2.2. We focus attention at the large pension system featuring $\theta = 0.025$. Comparing columns (b) and (f) we find that output per efficiency unit of labour drops by 1.74% ($\hat{y} = 9.668$) whilst the steady-state capital intensity falls by 6.19% ($\hat{k} = 0.350$). As a result of the decrease in the capital intensity, the annual interest rate rises by 12 basis points ($\hat{r}^a = 5.23\%$) whilst the wage rate falls by 1.74%. The proportion of constrained individual rises from 5.83% to 19.33%. The adverse selection index, as defined in (2.35) above, increases to $\widehat{AS} = 1.40$, the asset-weighted average survival rate of annuitants rises to $\hat{\bar{\mu}}^p = 0.70$, and the return on private annuities decreases slightly to $\hat{\bar{r}}^p = 9.98$. Finally, as the welfare indicators at the bottom of Table 2.2 reveal, under pension system B poor-health individuals are better off compared to the AI case as a result of the redistributionary feature of system B. The opposite holds for the healthy agents. Even though the policy maker cannot observe an individual's health status, by including a redistributionary component in the public pension system, the unhealthiest in society are aided somewhat.

In Figure 2.10 we present the η-averaged profiles for consumption during youth and old-age, annuity demand, and lifetime utility. These profiles are defined as:

$$\begin{aligned}
\frac{\hat{C}^y(\mu)}{\hat{w}} &= (1-\theta)\int_{\eta_L}^{\eta_H}\left[1-\mathbb{I}_{SB}(\mu,\eta)\right]\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad+\left(1-\theta+\frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right)\int_{\eta_L}^{\eta_H}\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad+\theta\frac{\hat{\bar{\mu}}^p}{\bar{\mu}}\int_{\eta_L}^{\eta_H}\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\zeta_B(\eta)\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta, \qquad (2.75)\\
\frac{\hat{C}^o(\mu)}{\hat{w}} &= \int_{\eta_L}^{\eta_H}\left(\lambda+\theta\zeta_B(\eta)\frac{1+\hat{r}}{\bar{\mu}}\right)\left[1-\mathbb{I}_{SB}(\mu,\eta)\right]\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad+\left((1-\theta)\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}+\lambda\right)\int_{\eta_L}^{\eta_H}\left[1-\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\right]\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad+\theta\frac{1+\hat{r}}{\bar{\mu}}\int_{\eta_L}^{\eta_H}\left[1-\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\right]\zeta_B(\eta)\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta, \qquad (2.76)\\
\frac{\hat{A}^p(\mu)}{\hat{w}} &= (1-\theta)\int_{\eta_L}^{\eta_H}\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad-\left(1-\theta+\frac{\lambda\hat{\bar{\mu}}^p}{1+\hat{r}}\right)\int_{\eta_L}^{\eta_H}\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta \\
&\quad-\theta\frac{\hat{\bar{\mu}}^p}{\bar{\mu}}\int_{\eta_L}^{\eta_H}\Phi\left(\mu,\frac{1+\hat{r}}{\hat{\bar{\mu}}^p}\right)\zeta_B(\eta)\mathbb{I}_{SB}(\mu,\eta)\frac{\eta h(\mu,\eta)}{h_\mu(\mu)}d\eta, \qquad (2.77)
\end{aligned}$$

where $\mathbb{I}_{SB}(\mu,\eta)=0$ if $\hat{A}^p(\mu,\eta)<0$ and $\mathbb{I}_{SB}(\mu,\eta)=1$ if $\hat{A}^p(\mu,\eta)\geq 0$.[8] The profiles for SB and SA (in Figure 2.6) are very similar.

2.4.3 Pension system C

The final case we consider is pension system C under which the government engages in more extreme redistribution from the rich to the poor (than under system B) by providing every surviving individual with the *same* pension payment:

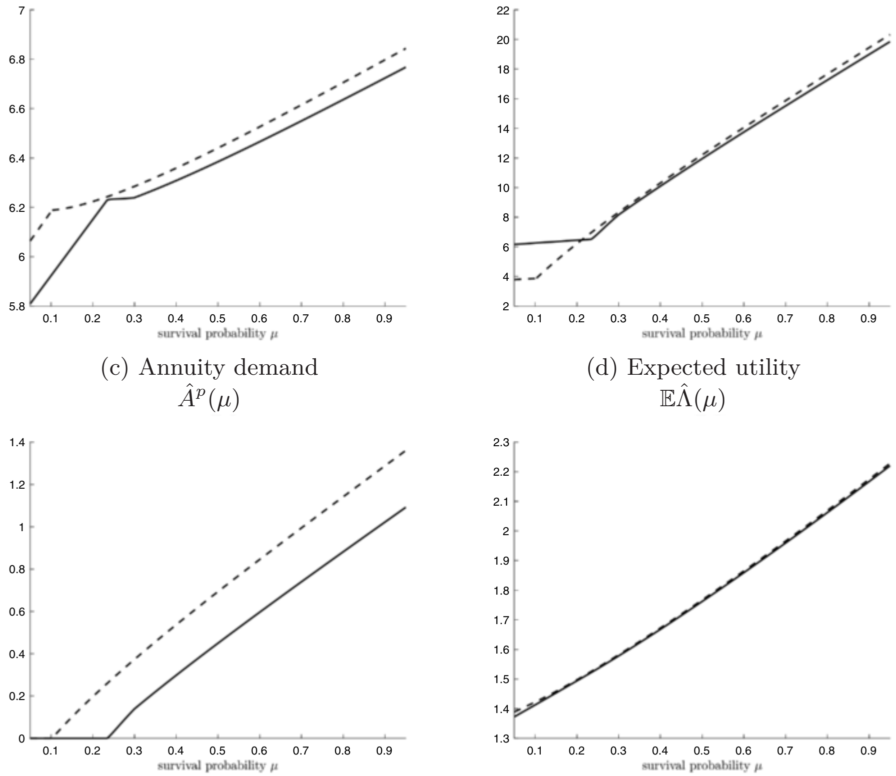
$$R^s_{t+1}(\eta)=\bar{R}^s_{t+1}, \qquad (2.78)$$

where $\bar{R}^s_{t+1}$ is to be determined below. The clearing condition for the public pension system is given in this case by:

$$(1+r_{t+1})\theta\bar{\eta}w_tL_t = L_t\int_{\mu_L}^{\mu_H}\int_{\eta_L}^{\eta_H}\mu\bar{R}^s_{t+1}h(\mu,\eta)d\eta d\mu, \qquad (2.79)$$

[8]Using Figure 2.7 the indicator function $\mathbb{I}_{SB}(\mu,\eta)$ can be characterized a bit further. For $\mu_L\leq\mu<\hat{\mu}^{bc}(\eta_H)$ all individuals are constrained, i.e. $\mathbb{I}_{SB}(\mu,\eta)=0$ for all $\eta\in[\eta_L,\eta_H]$. Similarly, for $\hat{\mu}^{bc}(\eta_L)\leq\mu<\mu_H$ all individuals are unconstrained, i.e. $\mathbb{I}_{SB}(\mu,\eta)=1$ for all $\eta\in[\eta_L,\eta_H]$. Finally, for $\hat{\mu}^{bc}(\eta_H)\leq\mu\leq\hat{\mu}^{bc}(\eta_L)$ we define the critical level of η at which borrowing constraints cease to bind, i.e. $\hat{\eta}^{bc}(\mu)$ is the inverse function of $\hat{\mu}^{bc}(\eta)$ in that domain. Then $\mathbb{I}_{SB}(\mu,\eta)=0$ for $\eta_L\leq\eta<\hat{\eta}^{bc}(\mu)$ and $\mathbb{I}_{SB}(\mu,\eta)=1$ for $\hat{\eta}^{bc}(\mu)\leq\eta\leq\eta_H$.

Figure 2.10: Steady-state profiles under pension system B



(a) Youth consumption $\hat{C}^y(\mu)$

(b) Old-age consumption $\hat{C}^o(\mu)$

(c) Annuity demand $\hat{A}^p(\mu)$

(d) Expected utility $\mathbb{E}\hat{\Lambda}(\mu)$

Legend The solid lines depict the steady-state profiles under pension system B (SB), and the dashed lines visualize the profiles for the asymmetric information (AI) case without pensions. In both cases adverse selection results in a single pooling rate of interest on annuities, $\bar{r}^p_{t+1}$, and agents with poor health face binding borrowing constraints. The SB case has been drawn for a large system featuring $\theta = 0.025$.

so that $\bar{R}^s_{t+1}$ is given by:

$$\bar{R}^s_{t+1} = \theta\bar{\eta}w_t\frac{1+r_{t+1}}{\bar{\mu}}. \tag{2.80}$$

Expressing the pension receipt in terms of the contribution made during youth we find for a person of type η that $R^s_{t+1}(\eta) = \zeta(\eta)\theta w_t(\eta)$ where $\zeta(\eta)$ is given by:

$$\zeta(\eta) = \zeta_C(\eta)\,\frac{1+r_{t+1}}{\bar{\mu}}, \qquad \zeta_C(\eta) \equiv \frac{\bar{\eta}}{\eta}. \tag{2.81}$$

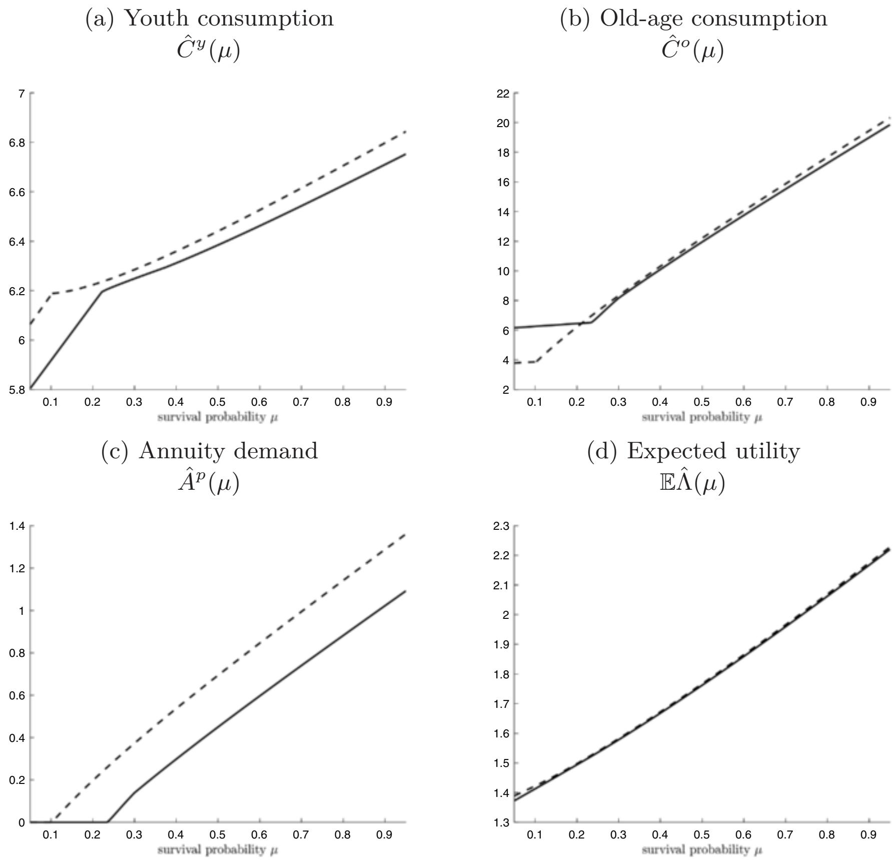
See Figure 2.9 for features of pension system C. It follows from (2.81) that for individuals with above-average productivity, $\eta > \bar{\eta}$, the rate of return on social annuities falls short of the actuarially fair social annuity yield, $(1 + r_{t+1})/\bar{\mu}$. In contrast, below-average individuals get a better-than actuarially fair rate on the pension contributions. Intuitively these results follow from the fact that the pension system redistributes resources from productive to less productive agents.

Qualitatively system C is very similar to system B (in that both feature redistribution for healthy to unhealthy agents) and the key expressions characterizing system C can be obtained by replacing $\zeta_B(\eta)$ with $\zeta_C(\eta)$ in equations (2.68)–(2.77). The main features of system C are the following. First, the comparison of columns (b) and (g) in Table 2.2 reveals that output and wages fall by 1.83% ($\hat{y} = 9.660$) and the capital intensity drops by 6.49% ($\hat{k} = 0.349$). Out of the three pension systems considered, system C features the largest macroeconomic effects. Redistribution is macroeconomically costly. Second, from Figure 2.9 it is clear that pension system C indeed features the highest degree of redistribution from healthy to unhealthy individuals. Indeed, as can be observed in panel (c) the effective contribution rate $\theta^n_t(\eta)$ becomes positive for the most healthy individuals. Such individuals experience the pension system as a tax burden. Third, as is shown in Figure 2.8 low-ability types are affected most severely by borrowing constraints under pension system C. Finally, the individual η-averaged profiles for consumption, annuity demands, and utility are depicted in Figure 2.11. These profiles are very similar to the ones we found for system B.

2.5 Privatizing social security

The key message of the previous section is loud and clear. The mandatory funded pension systems that we have studied are immune to adverse selection by design but they exacerbate the adverse selection problem in the market for private annuities, increase the fraction of borrowing-constrained ('over-annuitized') individuals in the population, and lead to long-run crowding out of capital and substantial output losses. This begs the following question: is it better to privatize social security altogether and to allow individuals to insure against longevity risk in the private annuity market even though this market is not perfect? Referring to Table 2.2 we find that abolishing the large pension system A (featuring $\theta = 0.025$) would increase output by 1.65% in the long run. In addition, it would increase steady-state welfare of all

Figure 2.11: Steady-state profiles under pension system C



Legend The solid lines depict the steady-state profiles under pension system C (SC), and the dashed lines visualize the profiles for the asymmetric information (AI) case without pensions. In both cases adverse selection results in a single pooling rate of interest on annuities, $\bar{r}^p_{t+1}$, and agents with poor health face binding borrowing constraints. The SC case has been drawn for a large system featuring $\theta = 0.025$.

corner types in the economy, cf. the information contained in columns (d) and (b). At least in the long run, privatization is a 'win-win' scenario.

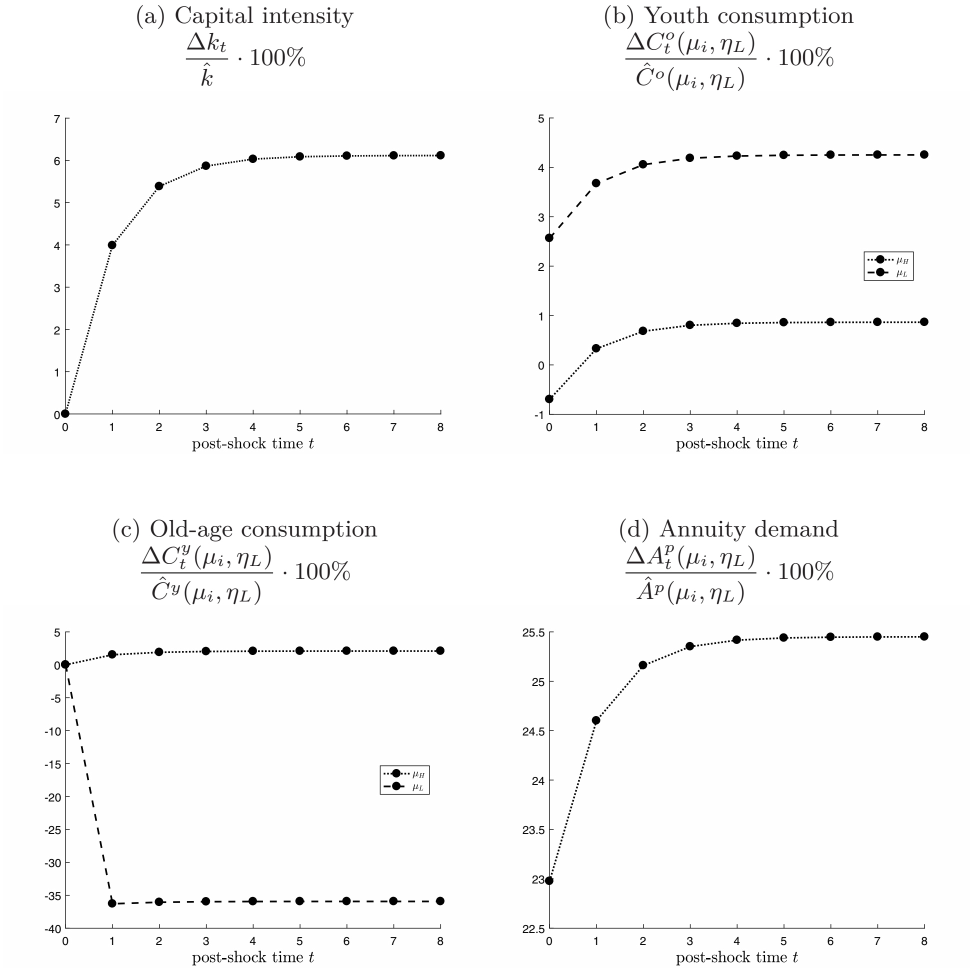
Of course, comparing steady states gives only part of the answer. What matters is whether or not is possible to abolish the funded pension system in a Pareto improving manner, i.e. is it a 'win-win' scenario to all generations? To answer this question we now study the transitional dynamic effects of abolishing pension system A. The economy is in the steady state for the SA system with $\theta = 0.025$ and the capital intensity is equal to $\hat{k}_{SA} = 0.351$. At shock-time $t = 0$, the pension system is abolished so that young individuals do not pay the pension contribution anymore, i.e. wage income from $t = 0$ onward equals $w_t(\eta)$ and pensions receipts from period $t = 1$ onward are equal to zero, $R_t^s(\eta) = 0$. Of course the old survivors at the time of the shock receive the pension they saved for, i.e. $R_0^s(\eta) > 0$.

Figure 2.12 depicts some of the key features of the transition process. Panel (a) shows that the capital intensity is predetermined at impact but thereafter rises monotonically to settle at the new steady-state level associated with the AI equilibrium, $\hat{k}_{AI} = 0.373$. Panel (b) show the percentage change in youth-consumption for healthy and unhealthy individuals with the lowest skill level. Interestingly, the healthy individuals decrease their consumption whilst the unhealthy increase it. The response of the latter group of people is easy to understand: these individuals were facing severe borrowing constraints in the SA system (and will continue to do so to a lesser degree in the AI equilibrium). Because the pension system is abolished (and $\theta = 0$) they can increase their consumption during youth and reduce the degree of overannuitization. Note that in panel (c) the overannuitization faced by the unhealthy is illustrated by the dramatic fall in old-age consumption for period $t = 1$ (when the surviving shock-time young are old) and beyond. Finally, in panel (d) we show that there is a strong increase in the demand for private annuities by the healthy agents.[9] There is virtually no transitional dynamics in μ_t^{bc} which falls from $\hat{\mu}^{bc} = 0.2090$ to $\mu_0^{bc} = 0.1026$ and thereafter settles at $\hat{\mu}^{bc} = 0.1025$. It follows that all agents featuring $\mu < 0.1025$ face borrowing constraints during youth no matter when they are born.

In Figure 2.13 we illustrate the effects on lifetime welfare for the four corner types in the economy, i.e. (μ_L, η_L), (μ_L, η_H), (μ_H, η_L), and (μ_H, η_H). Regardless of when they are born and irrespective of their productivity level, the unhealthiest individuals are better off as a result of the pension abolishment. Expected lifetime utility rises over time so for all corner types the gain is higher the later they are born. Interestingly, healthy agents born at the time of the shock are worse off than they would have been under the SA system. Privatizing social security is not a 'win-win' scenario to all generations.

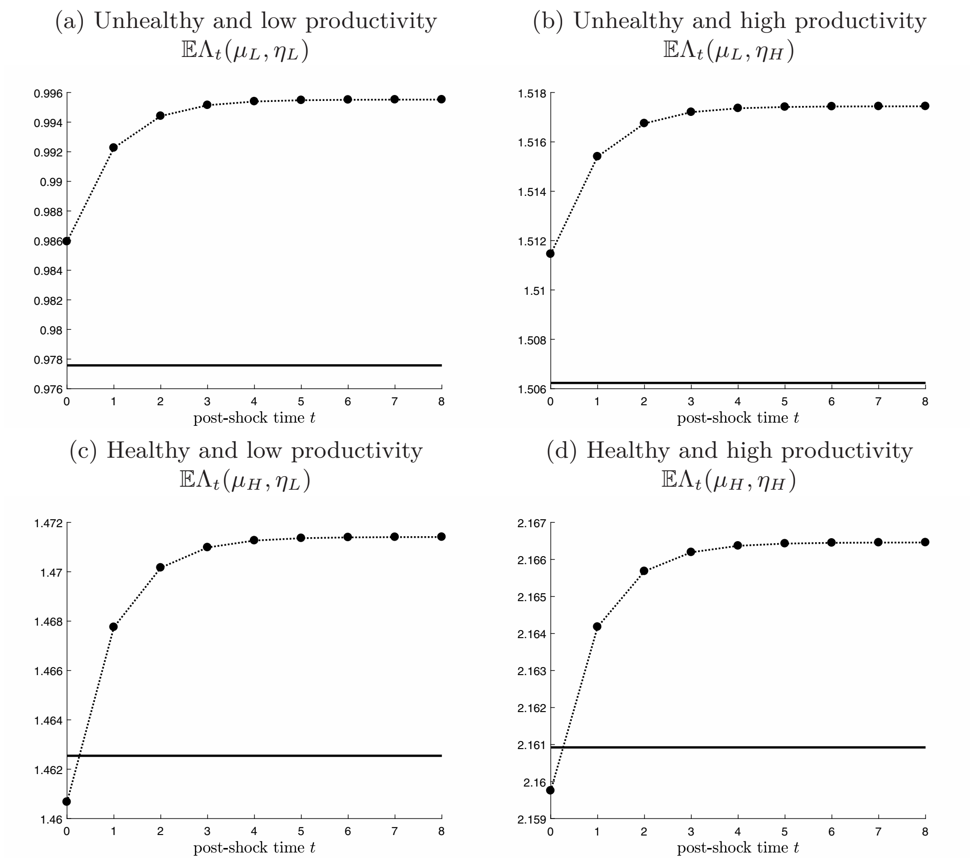
[9] Since youth-consumption, private annuity demand, and old-age consumption are linear in η for both SA and AI systems, it follows that the information in panels (b)-(d) is the same for all values of η.

Figure 2.12: Abolishing pension system A



Legend At time $t = 0$ pension system A with a contribution rate of $\theta = 0.025$ is abolished permanently. The system is initially in the steady state featuring a capital intensity $\hat{k}_{SA} = 0.351$. Panel (a): over time the economy converges monotonically to the steady-state for the AI case with $\hat{k}_{AI} = 0.373$. Panels (b)-(c) show the percentage change in, respectively, youth and old-age consumption for an individual of type (μ_i, η_L). Panel (d) depicts the percentage change in annuity demand of a person of type (μ_H, η_L). See also Table 2.2.

Figure 2.13: Lifetime utility of corner types



Legend The solid lines depict the steady-state lifetime utility levels attained by the different corner types under pension system A (SA) with $\theta = 0.025$. The abolishment of the pension system occurs at time $t = 0$ and affects lifetime utility of different types over time. Unhealthy agents benefit from the policy initiative no matter when they are born. Healthy individuals born at the time of the shock are worse off as a result of it.

2.6 Conclusion

In this Chapter we have developed an overlapping generations model which features adverse selection in the private annuity market and endogenously determined borrowing constraints in the capital market. Consumers are assumed to be heterogeneous in two dimensions—working ability and health status—which in the absence of perfect information leads to adverse selection in the private annuity market. Furthermore, they are restricted from borrowing against their anticipated future wage income due to the borrowing constraints. We demonstrate numerically that the informational asymmetry matters quantitatively in that, compared to the world with perfect information, it causes first-order reductions in output per efficiency unit of labour and the capital intensity. Starting from the benchmark model with adverse selection we introduce a fully-funded social security system and study its impact on capital accumulation and individual welfare under three different pension benefit rules.

We find that the social security system affects both capital accumulation and the proportion of individuals that are facing borrowing constraints. Capital crowding out increases and borrowing constraints become more prevalent the larger is the pension system. Intuitively a social security system causes more consumers to be over-annuitized and to face borrowing constraints. They cannot undo the effects of social security by transacting in their private accounts because any attempt to go short on annuities (demanding life-insured loans) would reveal their health status to the insurance companies in a world with asymmetric information.

The welfare effects of social security depend both on the pension recipient's type and on the specific form of the pension benefit rule. Provided the rule incorporates some implicit or explicit redistribution from healthy to unhealthy individuals, the latter group will actually benefit from the existence of the social security system in the steady state. In contrast, if pension benefits are proportional to an individual's contributions during youth and the proportionality factor is the same for everybody then the pension system makes everybody worse off in the long run.

A comparison of steady-state equilibria is not a guarantee that the privatization of social security is Pareto improving for all generations. For example, the simulations have shown that the abolition of a public pension system featuring a proportional benefit rule will harm shock-time healthy individuals. Even though all other generations and types are better off as a result, the privatization does not constitute a 'win-win' scenario.

In this Chapter we have intentionally ignored the role of an intentional bequest motive and its effect on capital accumulation. We are going to study these topics in later Chapters. Of course, the intention to leave bequests to one's offspring does affect an individual's attitude toward private annuities. Indeed, with an operative bequest motive, the rational individual will no longer fully annuitize his/her assets. Despite the high return on private annuities the individual will put aside a certain amount of unannuitized savings to pass on to their offspring upon death. In future work we intend to generalize the heterogeneous-agent model developed here by including an intentional bequest motive and to study the effects of social security with this extended framework.

2.A Appendix A

In this appendix we show some important results regarding the bivariate uniform distribution for μ and η that is employed in this Chapter (see equation (2.1) for the density function). First we show how to derive it by using the Farly-Morgenstern Family approach. In doing so we impose that the marginal distribution of μ (denoted by $h_\mu(\mu)$) is uniform in the interval $[\mu_L, \mu_H]$ whilst the one for η (denoted by $h_\eta(\eta)$) is uniform in the interval $[\eta_L, \eta_H]$. It follows that:

$$h_\mu(\mu) = \begin{cases} 1/(\mu_H - \mu_L) & \text{for } \mu_L \leq \mu \leq \mu_H \\ 0 & \text{otherwise,} \end{cases} \tag{A.1}$$

and:

$$h_\eta(\eta) = \begin{cases} 1/(\eta_H - \eta_L) & \text{for } \eta_L \leq \eta \leq \eta_H \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

For future reference we define the unconditional means as:

$$\bar{\mu} \equiv \int_{\mu_L}^{\mu_H} \mu h_\mu(\mu)\, d\mu = \frac{\mu_L + \mu_H}{2}, \qquad \bar{\eta} \equiv \int_{\eta_L}^{\eta_H} \eta h_\eta(\eta)\, d\eta = \frac{\eta_L + \eta_H}{2}. \tag{A.3}$$

Next we define the corresponding cumulative distribution functions as:

$$H_\mu(\mu) \equiv \int_{\mu_L}^{\mu} h_\mu(s) ds = \frac{\mu - \mu_L}{\mu_H - \mu_L}, \qquad H_\eta(\eta) \equiv \int_{\eta_L}^{\eta} h_\eta(s) ds = \frac{\eta - \eta_L}{\eta_H - \eta_L}. \tag{A.4}$$

Rice (2007, pp. 77-78) shows that for any parameter α such that $|\alpha| < 1$ a bivariate distribution $H(\mu, \eta)$ possessing uniform marginal distributions is obtained by computing:

$$H(\mu, \eta) = H_\mu(\mu) H_\eta(\eta) \left[1 + \alpha \left[1 - H_\mu(\mu)\right]\left[1 - H_\eta(\eta)\right]\right]. \tag{A.5}$$

Because $\lim_{\mu \to \mu_H} H_\mu(\mu) = 1$ and $\lim_{\eta \to \eta_H} H_\eta(\eta) = 1$ we find that the marginal distributions resulting from (A.5) are $H(\mu_H, \eta) = H_\eta(\eta)$ and $H(\mu, \eta_H) = H_\mu(\mu)$.

By using the expression from (A.4) in (A.5) we find that:

$$H(\mu, \eta) = \frac{(\mu - \mu_L)(\eta - \eta_L)}{(\mu_H - \mu_L)(\eta_H - \eta_L)} \left[1 + \alpha \frac{(\mu_H - \mu)(\eta_H - \eta)}{(\mu_H - \mu_L)(\eta_H - \eta_L)}\right]. \tag{A.6}$$

It follows from (A.6) that the density function, $h(\mu, \eta)$, is given by:

$$h(\mu, \eta) \equiv \frac{\partial^2 H(\mu, \eta)}{\partial \mu \partial \eta} = \frac{1 + \xi(\mu - \bar{\mu})(\eta - \bar{\eta})}{(\mu_H - \mu_L)(\eta_H - \eta_L)}, \tag{A.7}$$

where we have used the fact that $2\bar{\mu} = \mu_L + \mu_H$, $2\bar{\eta} \equiv \eta_L + \eta_H$, and define the parameter:

$$\xi \equiv \frac{4\alpha}{(\mu_H - \mu_L)(\eta_H - \eta_L)}. \tag{A.8}$$

The distribution function (A.6) can thus be written as:

$$H(\mu, \eta) = \frac{(\mu - \mu_L)(\eta - \eta_L)}{(\mu_H - \mu_L)(\eta_H - \eta_L)} \left[1 + \frac{\xi}{4} (\mu_H - \mu)(\eta_H - \eta)\right]. \tag{A.9}$$

Second, we compute some locational parameters for the bivariate uniform distribution. The unconditional means are stated in (A.3). For the unconditional variances we find:

$$\sigma_\mu^2 \equiv \operatorname{var}(\mu) \equiv \mathrm{E}\left[\mu - \bar{\mu}\right]^2 = \int_{\mu_L}^{\mu_H} \mu^2 h_\mu(\mu)\, d\mu - \bar{\mu}^2 = \frac{(\mu_H - \mu_L)^2}{12}, \tag{A.10}$$

$$\sigma_\eta^2 \equiv \operatorname{var}(\eta) \equiv \mathrm{E}\left[\eta - \bar{\eta}\right]^2 = \int_{\eta_L}^{\eta_H} \eta^2 h_\eta(\eta)\, d\eta - \bar{\eta}^2 = \frac{(\eta_H - \eta_L)^2}{12}. \tag{A.11}$$

The following lemma is useful.

Lemma A.1. *The following density functions can be derived:*

$$\Gamma_1(\mu) \equiv \frac{\int_{\eta_L}^{\eta_H} \eta h(\eta, \mu)\, d\eta}{\int_{\eta_L}^{\eta_H} h(\eta, \mu)\, d\eta} = \bar{\eta} + \xi\sigma_\eta^2(\mu - \bar{\mu}),$$

$$\Gamma_2(\eta) \equiv \frac{\int_{\mu_L}^{\mu_H} \mu h(\eta, \mu)\, d\mu}{\int_{\mu_L}^{\mu_H} h(\eta, \mu)\, d\mu} = \bar{\mu} + \xi\sigma_\mu^2(\eta - \bar{\eta}).$$

Proof The derivation of proceeds as follows.

$$\begin{aligned}
\Gamma_1(\mu) &\equiv \frac{\int_{\eta_L}^{\eta_H} \eta h(\eta, \mu)\, d\eta}{h_\mu(\mu)} \\
&= \frac{1}{\eta_H - \eta_L} \int_{\eta_L}^{\eta_H} \left[[1 - \xi(\mu - \bar{\mu})\bar{\eta}]\eta + \xi(\mu - \bar{\mu})\eta^2 \right] d\eta \\
&= \frac{1}{\eta_H - \eta_L} \left[[1 - \xi(\mu - \bar{\mu})\bar{\eta}] \frac{\eta_H^2 - \eta_L^2}{2} + \xi(\mu - \bar{\mu}) \frac{\eta_H^3 - \eta_L^3}{3} \right].
\end{aligned} \tag{A.12}$$

Note that for $\chi = \eta$ or $\chi = \mu$ we can write:

$$\begin{aligned}
\chi_H^2 - \chi_L^2 &= (\chi_H - \chi_L)(\chi_H + \chi_L) = 2\bar{\chi}(\chi_H - \chi_L), \\
\chi_H^3 - \chi_L^3 &= (\chi_H - \chi_L)\left(\chi_H^2 + \chi_L\chi_H + \chi_L^2\right) = (\chi_H - \chi_L)\left[(2\bar{\chi})^2 - \chi_L\chi_H\right],
\end{aligned} \tag{A.13}$$

$$\bar{\chi}^2 - \chi_L\chi_H = \frac{(\chi_H - \chi_L)^2}{4}.$$

Using these results for $\chi = \eta$, the term in square brackets on the right-hand side of

(A.12) can be simplified to:

$$\begin{aligned}
[\cdot] &= [1-\xi(\mu-\bar{\mu})\bar{\eta}]\bar{\eta}\,(\eta_H-\eta_L)+\xi(\mu-\bar{\mu})\frac{(\eta_H-\eta_L)\left[(2\bar{\eta})^2-\eta_L\eta_H\right]}{3}\\
&= (\eta_H-\eta_L)\left[[1-\xi(\mu-\bar{\mu})\bar{\eta}]\bar{\eta}+\frac{\xi(\mu-\bar{\mu})\left[4\bar{\eta}^2-\eta_L\eta_H\right]}{3}\right]\\
&= (\eta_H-\eta_L)\left[\bar{\eta}+\frac{\xi(\mu-\bar{\mu})\left[\bar{\eta}^2-\eta_L\eta_H\right]}{3}\right]\\
&= (\eta_H-\eta_L)\left[\bar{\eta}+\frac{\xi(\mu-\bar{\mu})\,(\eta_H-\eta_L)^2}{12}\right].
\end{aligned} \tag{A.14}$$

By using (A.14) and noting the definition of σ_η^2 in (A.12) we obtain the result to be proved. The derivation for Γ_2 proceeds along similar lines. ■

After some straightforward but tedious manipulations we find:

$$\begin{aligned}
\mathrm{E}\,(\mu\eta) &= \int_{\mu_L}^{\mu_H}\mu\eta h(\mu,\eta)d\mu\\
&= \int_{\mu_L}^{\mu_H}\mu\frac{\bar{\eta}+\xi\sigma_\eta^2(\mu-\bar{\mu})}{\mu_H-\mu_L}d\mu\\
&= \frac{1}{\mu_H-\mu_L}\left[(\bar{\eta}-\xi\sigma_\eta^2\bar{\mu})\frac{\mu_H^2-\mu_L^2}{2}+\xi\sigma_\eta^2\frac{\mu_H^3-\mu_L^3}{3}\right]\\
&= (\bar{\eta}-\xi\sigma_\eta^2\bar{\mu})\bar{\mu}+\xi\sigma_\eta^2\frac{(2\bar{\mu})^2-\mu_H\mu_L}{3}\\
&= \bar{\mu}\bar{\eta}+\xi\sigma_\eta^2\frac{\bar{\mu}^2-\mu_H\mu_L}{3}\\
&= \bar{\mu}\bar{\eta}+\xi\sigma_\mu^2\sigma_\eta^2,
\end{aligned} \tag{A.15}$$

where we have used the results from (A.13) for $\chi=\mu$. Hence, the covariance between η and μ is given by:

$$\mathrm{cov}\,(\mu,\eta)\equiv\mathrm{E}\,(\mu\eta)-\bar{\mu}\bar{\eta}=\xi\sigma_\mu^2\sigma_\eta^2, \tag{A.16}$$

and the correlation coefficient between η and μ is:

$$\mathrm{cor}\,(\mu,\eta)\equiv\frac{\mathrm{cov}\,(\mu,\eta)}{\sqrt{\mathrm{var}\,(\mu)\,\mathrm{var}\,(\eta)}}=\xi\sigma_\mu\sigma_\eta. \tag{A.17}$$

If $\xi=0$ then μ and η are uncorrelated.

2.B Appendix B

In the presence of binding borrowing constraints, $\mu_L<\mu_t^{bc}<\mu_H$, we define the average survival rate of annuitants by:

$$\bar{\mu}_t^{an}\equiv\frac{\int_{\mu_t^{bc}}^{\mu_H}\mu h_\mu(\mu)d\mu}{\int_{\mu_t^{bc}}^{\mu_H}h_\mu(\mu)d\mu}.$$

In this appendix we prove that $\bar{\mu}_t^p > \bar{\mu}_t^{an} > \bar{\mu}$ and $AS_t > 1$. The proof of $\bar{\mu}_t^{an} > \bar{\mu}$ is obvious. To show that $\bar{\mu}_t^p > \bar{\mu}_t^{an}$ is less trivial. The proof for this result proceeds along the lines of Heijdra and Reijnders (2012, fn. 7). Individual annuity demand can be written in a separable form as:

$$A_t^p(\mu, \eta) \equiv A_t^p(\mu)\eta,$$

with:

$$A_t^p(\mu) \equiv 1 - \Phi\left(\mu, \frac{1 + r_{t+1}}{\bar{\mu}_t^p}\right)\left[w_t + \frac{\lambda \bar{\mu}_t^p w_{t+1}}{1 + r_{t+1}}\right].$$

Since $\partial\Phi(\cdot)/\partial\mu < 0$ it follows readily that $\partial A_t^p(\mu)/\partial\mu > 0$. The expression for the pooling rate can be rewritten as:

$$\bar{\mu}_t^p \int_{\mu_t^{bc}}^{\mu_H} Z_t(\mu) h_\mu(\mu) d\mu = \int_{\mu_t^{bc}}^{\mu_H} Z_t(\mu)\mu h_\mu(\mu) d\mu,$$

where $Z_t(\mu)$ is defined as follows:

$$Z_t(\mu) \equiv A_t^p(\mu)\Gamma_1(\mu).$$

Note that $Z_t(\mu)$ is increasing in μ (because both $A_t^p(\mu)$ and $\Gamma_1(\mu)$ are) so that $\text{cov}(Z_t(\mu), \mu) > 0$. Define the following average:

$$\bar{Z}_t \equiv \frac{\int_{\mu_t^{bc}}^{\mu_H} Z_t(\mu) h_\mu(\mu) d\mu}{\int_{\mu_t^{bc}}^{\mu_H} h_\mu(\mu) d\mu}.$$

By definition we have:

$$\begin{aligned}
\int_{\mu_t^{bc}}^{\mu_H} Z_t(\mu) h_\mu(\mu) d\mu &= \int_{\mu_t^{bc}}^{\mu_H} \bar{Z}_t h_\mu(\mu) d\mu \\
\int_{\mu_t^{bc}}^{\mu_H} Z_t(\mu)\mu h_\mu(\mu) d\mu &= \int_{\mu_t^{bc}}^{\mu_H} \left[Z_t(\mu) - \bar{Z}_t + \bar{Z}_t\right]\left[\mu - \bar{\mu}_t^{an} + \bar{\mu}_t^{an}\right] h_\mu(\mu) d\mu \\
&= \bar{\mu}_t^{an} \int_{\mu_t^{bc}}^{\mu_H} \bar{Z}_t h_\mu(\mu) d\mu + \text{cov}(Z_t(\mu), \mu)
\end{aligned}$$

It follows that $\bar{\mu}_t^p$ can be written as:

$$\bar{\mu}_t^p = \bar{\mu}_t^{an} + \frac{\text{cov}(Z_t(\mu), \mu)}{\int_{\mu_t^{bc}}^{\mu_H} \bar{Z}_t h_\mu(\mu) d\mu} > \bar{\mu}_t^{an},$$

where the inequality follows from the fact that $\text{cov}(Z_t(\mu), \mu) > 0$. Hence we have established that $\bar{\mu}_t^p > \bar{\mu}_t^{an} > \bar{\mu}$ and thus $AS_t > 1$. ■

2.C Appendix C

In this appendix we briefly discuss the micro- and macroeconomic effects of public pensions under perfect information. This was also the subject matter of Abel (1987).

In our discussion we focus the attention on pension system A. We conclude this appendix with a brief evaluation of the quantitative results for systems B and C.

Under pension system A the income endowment points are given by:

$$(1-\theta)\eta w_t, \qquad \lambda\eta w_{t+1} + \theta\zeta_A \frac{1+r_{t+1}}{\bar{\mu}}\eta w_t. \tag{C.1}$$

For given factor prices the endowments are linear in η. In terms of Figure C.1, the income endowment line IEL is a ray from the origin and individuals are distributed on the line segment $\mathrm{E}_L\mathrm{E}_H$. Note that IEL is a counter-clockwise rotation of the income endowment line without pensions (the dashed line in the figure). The budget constraints of an individual with characteristics μ and η are given by:

$$C_t^y(\mu,\eta) + A_t^p(\mu,\eta) = (1-\theta)\eta w_t, \tag{C.2}$$

$$C_{t+1}^o(\mu,\eta) = \lambda\eta w_{t+1} + \frac{1+r_{t+1}}{\mu}A_t^p(\mu,\eta) + \theta\zeta_A \frac{1+r_{t+1}}{\bar{\mu}}\eta w_t, \tag{C.3}$$

where we have used the expression for the full-information annuity rate of interest from (2.4). Under full information there is no sign restriction on annuity demand. Indeed, if an individual chooses $A_t^p(\mu,\eta) < 0$ then he/she purchases a life-insured loan (at the actuarially fair borrowing rate). As a result the lifetime budget constraint of the individual is:

$$C_t^y(\mu,\eta) + \frac{\mu C_{t+1}^o(\mu,\eta)}{1+r_{t+1}} = \left[1-\theta+\theta\zeta_A\frac{\mu}{\bar{\mu}}\right]\eta w_t + \frac{\lambda\eta\mu w_{t+1}}{1+r_{t+1}}. \tag{C.4}$$

The effective pension contribution rate is defined as:

$$\theta_t^n \equiv \theta\left(1-\zeta_A\frac{\mu}{\bar{\mu}}\right) \gtreqless 0 \quad \text{for} \quad \mu \lesseqgtr \frac{\bar{\mu}}{\zeta_A}. \tag{C.5}$$

Note that θ_t^n gets larger the unhealthier is the individual so that—in stark contrast to the asymmetric information case—the unhealthiest individuals actually face a positive contribution rate. The public pension system provides such individuals with a highly disadvantageous social annuity rate based on the average survival rate in the population (rather than their own). Under full information, pension system A thus redistributes resources from unhealthy to healthy individuals.

Optimal consumption (during youth and old age) and private annuity demand are given by:

$$C_t^y(\mu,\eta) = \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\left[\left[1-\theta+\theta\zeta_A\frac{\mu}{\bar{\mu}}\right]w_t + \frac{\lambda\mu w_{t+1}}{1+r_{t+1}}\right]\eta, \tag{C.6}$$

$$\frac{\mu C_{t+1}^o(\mu,\eta)}{1+r_{t+1}} = \left[1-\Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\right]\left[\left[1-\theta+\theta\zeta_A\frac{\mu}{\bar{\mu}}\right]w_t + \frac{\lambda\mu w_{t+1}}{1+r_{t+1}}\right]\eta, \tag{C.7}$$

$$A_t^p(\mu,\eta) = (1-\theta)\eta w_t - \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\left[(1-\theta)w_t + \theta\zeta_A\frac{\mu w_t}{\bar{\mu}} + \frac{\lambda\mu w_{t+1}}{1+r_{t+1}}\right]\eta. \tag{C.8}$$

In Figure C.1 the choices of the four corner types are illustrated. For a pension system of realistic size, IEL lies to the right of MRSC and all individuals purchase private annuities, i.e. the nation's capital stock is not fully owned by the public pension system. For the lowest-ability individuals consumption occurs at points A (for μ_L) and B (for μ_H). For the highest-ability agents the consumption points are at C (for μ_L) and D (for μ_H). Hence, the introduction of a pension system A of realistic size does not cause the market for life-insured loans to become active. (For a large pension system, all agents buy life-insured loans and the national capital stock is owned by the public pension system.)

The fundamental difference equation for the capital intensity is given by:

$$k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\left[\theta\bar{\eta}w_t + \int_{\eta_L}^{\eta_H}\int_{\mu_L}^{\mu_H} A_t^p(\mu,\eta)h(\mu,\eta)d\mu d\eta\right]. \tag{C.9}$$

By substituting (C.8) into (C.9) and simplifying we find:

$$\begin{aligned} k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\Bigg[\bar{\eta}w_t - \int_{\mu_L}^{\mu_H}\left[(1-\theta)w_t + \theta\zeta_A\frac{\mu w_t}{\bar{\mu}} + \frac{\lambda\mu w_{t+1}}{1+r_{t+1}}\right] \\ \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)h_\mu(\mu)\Gamma_1(\mu)d\mu\Bigg], \end{aligned} \tag{C.10}$$

where factor prices follow from (2.20)–(2.21).

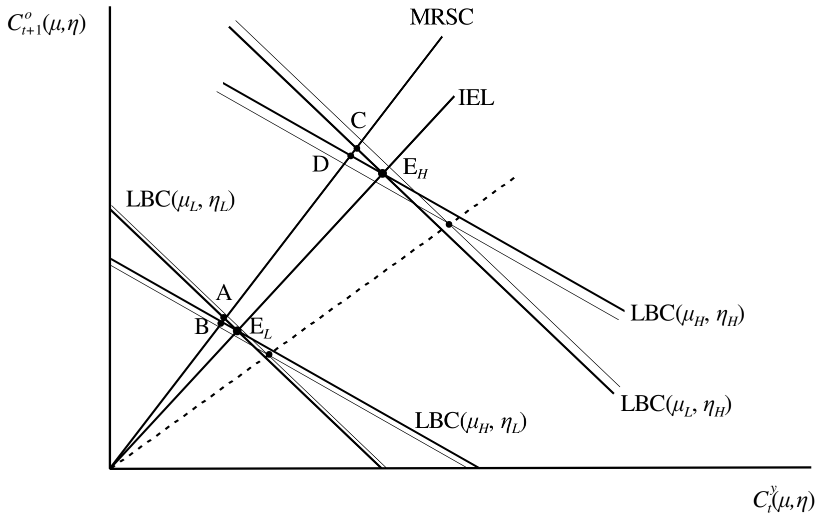
In column (b) of Table C.1 we report some key features of the steady-state equilibrium under pension system A. Comparing columns (b) and (a) we find that output per efficiency unit of labour and the wage rate both increase slightly (by 0.28%) whilst the steady-state capital intensity is increased somewhat (by 1.04%). As a result of the increase in the capital intensity, the annual interest rate falls by 2 basis points ($\hat{r}^a = 4.98\%$). Finally, as the welfare indicators at the bottom of Table C.1 reveal, under pension system A healthy (unhealthy) individuals are better (worse) off compared to the FI case. The pension system slightly stimulates capital accumulation but redistributes resources from unhealthy to healthy individuals.

In columns (c) and (d) of Table C.1 we also report the quantitative results for, respectively, pension systems B and C. The analytical expressions characterizing these equilibria are obtained by replacing ζ_A in (C.1)–(C.9) by, respectively, $\zeta_B(\eta)$ and $\zeta_C(\eta)$ given in (2.67) and (2.81) above. The expression for k_{t+1} is slightly more complicated because annuity demand is no longer linear in η under systems B and C:

$$\begin{aligned} k_{t+1} = \frac{1}{1+n}\frac{L_t}{N_t}\Bigg[\bar{\eta}w_t - \int_{\eta_L}^{\eta_H}\int_{\mu_L}^{\mu_H}\left[(1-\theta)w_t + \theta\zeta_i(\eta)\frac{\mu w_t}{\bar{\mu}} + \frac{\lambda\mu w_{t+1}}{1+r_{t+1}}\right] \\ \Phi\left(\mu, \frac{1+r_{t+1}}{\mu}\right)\eta h(\mu,\eta)d\mu d\eta\Bigg], \end{aligned} \tag{C.11}$$

for $i \in \{B, C\}$. Despite the fact that systems B and C incorporate some (implicit or explicit) redistribution from healthy to unhealthy individuals, the unhealthy continue to be worse off under both systems when compared to the FI case without pensions.

Figure C.1: Consumption-saving choices under full information and pension system A



Legend LBC(μ_i, η_j) is the lifetime budget constraint for an individual with survival probability μ_i and productivity level η_j. The thin lines represent the FI case without pensions. IEL is the income endowment line and agents are located on the line segment $E_L E_H$. MRSC is the consumption Euler equation under perfect information with actuarially fair annuities at the individual level. Optimal consumption for individual (μ_i, η_j) is located at the intersection of MRSC and LBC(μ_i, η_j). Provided the pension system is of a realistic size, IEL lies to the right of MRSC and all individuals purchase private annuities.

Table C.1: Pensions under full information

	(a) FI	(b) SA_A $\theta = 0.025$	(c) SA_B $\theta = 0.025$	(d) SA_C $\theta = 0.025$
$\hat{y}$	10.000	10.028	10.027	10.025
$\hat{k}$	0.395	0.400	0.399	0.399
%Q1	12.34	9.25	8.47	7.55
%Q2	19.81	14.84	14.37	14.07
%Q3	28.73	21.53	21.68	21.99
%Q4	39.12	29.31	30.41	31.31
%SAS		25.06	25.08	25.09
$\hat{r}$	6.04	5.99	5.99	5.99
$\hat{r}^a$	5.00%	4.98%	4.98%	4.98%
$\hat{w}$	7.250	7.271	7.269	7.268
$\widehat{BC}$	0.00%	0.00%	0.00%	0.00%
$\hat{c}^y$	5.357	5.368	5.367	5.367
%Q1	15.99	15.90	15.98	16.07
%Q2	22.10	22.05	22.10	22.13
%Q3	28.06	28.07	28.06	28.02
%Q4	33.85	33.98	33.86	33.76
$\hat{c}^o$	4.087	4.099	4.099	4.098
%Q1	12.23	12.17	12.26	12.37
%Q2	19.74	19.70	19.77	19.81
%Q3	28.75	28.76	28.75	28.72
%Q4	39.28	39.37	39.22	39.10
$\mathbb{E}\hat{\Lambda}(\mu_L, \eta_L)$	1.014	1.002	1.003	1.004
$\mathbb{E}\hat{\Lambda}(\mu_H, \eta_L)$	1.433	1.450	1.463	1.484
$\mathbb{E}\hat{\Lambda}(\mu_L, \eta_H)$	1.529	1.522	1.521	1.521
$\mathbb{E}\hat{\Lambda}(\mu_H, \eta_H)$	2.143	2.153	2.149	2.146

Note Here %Qj denotes the share accounted for by skill quartile j (averaged over all survival rates) of the variable directly above it. %SAS is the share owned by the social annuity system. $\mathbb{E}\hat{\Lambda}(\mu_i, \eta_j)$ gives expected utility for an agent with health type μ_i and skill type η_i.

Chapter 3

Annuities, Bequests and Asymmetric Information*

*This chapter is based on Heijdra, Jiang and Mierau (2017a).

3.1 Introduction

In a seminal contribution, Yaari (1965) showed that, in the absence of bequest motives, individuals facing life-time uncertainty should fully annuitize all their assets. Allowing for bequest motives, Davidoff *et al.* (2005) show that individuals should annuitize the part of their savings that they wish to consume in old age, whilst leaving the remainder in a non-annuitized savings account. However, in spite of their strong theoretical benefits, in practice annuity uptake is much lower – with individuals opting to hold the lion's share of their assets in non-annuitized accounts (see, for instance, Inkmann *et al.* (2011)).

The seeming discrepancy between the theoretical predictions by Yaari (1965) and Davidoff *et al.* (2005) and the empirical findings of, for instance, Inkmann *et al.* (2011) is commonly dubbed as the *annuity puzzle* and explanations for it have been sought both in the rational and the behavioral domain (see, Brown (2007) for a review).[1] As regards the behavioral domain, explanations for the puzzle have been sought in concerns about financial literacy and its associated cognitive constraints (Brown *et al.*, 2015). Within the rational domain the focus has been predominantly on market imperfections such as asymmetric information and adverse selection. Intuitively, such imperfections arise from the fact that in order for the annuity premium to be actuarial fair, it needs to be priced according to the survival probability of the individual. Such information is, however, generally unverifiable leading to asymmetric information between annuity firms and their (potential) customers with mis-priced annuities as a consequence. Mitchell *et al.* (1999), for instance, show that asymmetric information and adverse selection lead annuities to be priced substantially lower than in frictionless markets.

With the above in mind, Davidoff *et al.* (2005) use a two-period life-cycle model to show that as long as the annuity premium is higher than the return on non-annuitized savings accounts, individuals should annuitize all assets that they wish to consume in old age – regardless of whether the annuity premium has been driven down due to asymmetric information and adverse selection. Therefore, asymmetric information alone cannot be used as an explanation of the annuity puzzle. Hence, in this paper we combine two commonly considered explanations of the annuity puzzle – bequest motives and asymmetric information – to assess how their interplay can rationalize (some of) the annuity puzzle.

The starting point of our analysis is the celebrated paper by Abel (1986) in which he considers a two-period life-cycle model in which individuals differ according to their survival probabilities and shows that private annuities cannot perfectly substitute for a mandatory fully-funded social security system due to adverse selection caused by asymmetric information concerning individual mortality. Importantly, our analysis differs from his in the sense that he limits his analysis to individuals with

[1] In the current context we interpret the annuity puzzle as meaning that in reality individuals hold their assets mainly in non-annuitized savings, whilst existing theory would predict that at least a substantial share of their savings are annuitized.

a strictly positive demand for annuities. In contrast, we impose a borrowing constraints, which implies that individuals can hold a non-negative amount of annuities. This generalization of Abel's framework enables us to consider how the annuity puzzle may arise through combination of asymmetric information and bequest motives.

The main finding of this Chapter is that while bequest motives and asymmetric information in isolation cannot account for the annuity puzzle, their interplay can. Indeed, we find that the combination of asymmetric information and bequest motives will cause low-health (that is, high-mortality) individuals to refrain from annuitizing their assets. Intuitively, this result arises because bequest motives enhance the value of non-annuitized savings so that in order for individuals to choose annuities over non-annuitized savings they require annuities to be priced nearly actuarially fair. This is, however, not possible due to the pooling of health types as a consequence of asymmetric information.

To separate the impact of asymmetric information from that of adverse selection we derive our result in two steps. First, we consider the impact of having a publicly sponsored annuity plan in which the government offers an annuity premium based on the average mortality rate in the society. Here we show that for such a rate low-health individuals will already refrain from annuitizing their assets for the reasons outlined above. In the second step, we consider a private annuity market in which profit maximizing firms take into account that in order not to suffer losses due to low-health types dropping out of the annuity market, the market annuity rate has to be lower than the one based on the average mortality rate. Needless to say, this lower rate leads ever healthier individuals to retreat from the annuity market. Importantly, this decomposition of our main result implies that non-annuitization by low-health individuals cannot be solved by a government policy aimed at eliminating the adverse selection component of annuity prices.[2]

Through two extensions of our model we consider the interplay between bequest and asymmetric information for non-annuitization further. First of all, we consider the role of a pay-as-you-go social security system – a common program for financing retirement consumption. Here we show that such a system leads to crowding out of savings; as is to be expected. In addition, however, in the context of a private annuity market it also leads ever healthier individuals to retreat from the annuity market. Because social security is non-bequeathable we also observe that healthy individuals reduce the share of annuities in their retirement savings portfolio substantially. Second, taking cue from the literature on the health-wealth nexus (see, Currie (2009)), we introduce a correlation between health and earning ability of individuals. Here we show that such correlations will aggravate adverse selection on the annuity market as the higher wealth holdings of healthier individuals will push down the annuity premium more than if such correlations are absent.

This Chapter extends and adds to the extensive literature aimed at understanding the role of annuities for the provision of retirement income. Within this vast

[2]Elsewhere we consider whether the government should stimulate use of annuity markets in the first place, see Heijdra *et al.* (2014, 2016).

literature our particular focus is on the role of the interplay between bequests and asymmetric information. The cross-section between these items was pioneered by Abel (1986) and recently revived by Lockwood (2012) who combines bequest motives with the omnibus imperfection factor of Hansen and İmrohoroğlu (2008) and Heijdra and Mierau (2012) to show that generic annuity market imperfections in combination with bequest motives can give rise to the low observed levels of annuitization suggested by the analysis of, for instance, Inkmann *et al.* (2011). While following Mitchell *et al.* (1999), Lockwood attributes his imperfection factor to sources such as asymmetric information, he does not model the underlying heterogeneity between individuals as such. Hence, we open up the black box of Lockwood's generic imperfection factor by highlighting how it may arise and how its impact differs across health types.

The remainder of this Chapter is set up as follows. The next section outlines the analytical framework. Section 3.3 details the interplay between bequest motives and asymmetric information. The subsequent section contains extensions to a pay-as-you-go social security system and the health-wealth nexus. The final section concludes.

3.2 Analytical Framework

The basic set up of our model is similar to Abel (1986). Individuals live for two periods – youth and old age – and face uncertain survival from the first period to the next. While each individual knows his/her own survival probability, μ, with certainty it is not verifiable to outsiders. Individuals differ from each other in their survival probabilities and we assume that health types are continuously distributed from those with lowest survival probability, μ_L, to those with the highest probability of survival, μ_H according to some distribution function with probability density $h(\mu)$.[3]

During youth individuals inelastically supply one unit of labor and during old-age (s)he is retired. As a consequence (s)he earns a wage, w, in the first period, which is split between consumption, C^Y, non-annuitized savings, S, and annuities, A. Hence, the first period budget constraint equals:

$$C^Y + S + A = w. \tag{3.2.1}$$

Individuals have their offspring at the beginning of the second period, after which life-time uncertainty is resolved. If the individual dies, all non-annuitized savings, including any returns, flow to the offspring in the form of unintended bequests, B^U:

$$B^U = (1 + r)S, \tag{3.2.2}$$

where r is the interest rate on savings. By contrast, surviving individuals split their wealth between consumption, C^O, and intentional bequests, B^I:

$$C^O + B^I = (1 + r)S + (1 + r^A(\mu))A, \tag{3.2.3}$$

[3]Observe that by using Abel (1986) as a starting point our model generalizes that of Lockwood (2012) who abstracts from heterogeneity in individual health types and the consequent impact of asymmetric information and adverse selection.

where r^A is the return on annuities, which depends on the survival probability and may or may not be actuarially fair (see below). Combining (3.2.1)-(3.2.3) and imposing a borrowing constraint (*i.e.*, $A \geq 0$ & $S \geq 0$) provides the consolidated life-time intertemporal budget constraint:[4]

$$C^O + B^I = (1 + r^A(\mu))(w - C^Y) - \frac{r^A(\mu) - r}{1+r} B^A. \tag{3.2.4}$$

Individuals derive utility from own consumption as well as from the bequest left to their offspring. Hence, the expected life-time utility function of an individual is given by:

$$\mathbb{E}\Lambda = U(C^Y) + \frac{1-\mu}{1+\rho} V(B^U) + \frac{\mu}{1+\rho} \left(U(C^O) + V(B^I) \right), \tag{3.2.5}$$

where ρ is the pure rate of time preference and $U(C)$ and $V(B)$ are increasing and concave utility functions of own consumption and bequests, respectively. Regarding the utility function we follow much of the literature and use an iso-elastic function with an intertemporal elasticity of substitution equal to $\sigma \in (0, 1]$:

$$U(C) = \begin{cases} \frac{C^{1-1/\sigma} - 1}{1 - 1/\sigma} & \text{if} \quad \sigma > 0, \quad \sigma \neq 1, \\ \ln C & \text{if} \quad \sigma = 1. \end{cases} \tag{3.2.6}$$

In symmetry with the utility function, we assume that the bequest function also takes an iso-elastic form:

$$V(B) = \begin{cases} \eta \frac{(\theta + B)^{1-1/\sigma} - 1}{1 - 1/\sigma} & \text{if} \quad \sigma > 0, \quad \sigma \neq 1, \\ \eta \ln(\theta + B) & \text{if} \quad \sigma = 1, \end{cases} \tag{3.2.7}$$

where we follow the extant literature and let the σ be equal across the different sources of utility. The remaining structure is as follows: $\eta \geq 0$ describes the strength of the bequest motive and $\theta \geq 0$ determines the threshold wealth below which individuals do not leave bequests to their offspring. The structure in (3.2.7) can accommodate the most commonly used functional forms in the literature.[5] Lockwood (2012), for instance, assumes $\theta > 0$, which makes bequests luxury goods. Abel (1986), by contrast, lets $\theta = 0$ implying that each individual, regardless of his/her asset position, leaves the same proportion of wealth as bequests, which is at odds with empirical evidence (see, De Nardi *et al.* (2010) and Lockwood (2012)).

The objective of the individual is to maximize (3.2.5) subject to (3.2.4) and non-negativity constraints on bequests, savings and annuities leading to the following

[4]For sake of internal consistency, individuals who leave bequests should also receive bequests. Abel (1986) handles this by endowing each individual with some level of initial wealth. However, this only affects the implications of the model if a general equilibrium perspective is taken, which we do not pursue. In a partial equilibrium setting it becomes an exogenous constant of the maximization problem with no impact on the optimality conditions.

[5]See, Lockwood (2012) and Pashchenko (2013) for overviews.

first order conditions:

$$U'(C^Y) - (1 + r^A(\mu))\frac{\mu}{1+\rho}U'(C^O) = 0, \tag{3.2.8a}$$

$$U'(C^O) - V'(B^I) = 0, \tag{3.2.8b}$$

$$V'(B^U) - \frac{r^A(\mu) - r}{1+r}\frac{\mu}{1-\mu}V'(B^I) = 0, \tag{3.2.8c}$$

where we have assumed an interior solution (i.e., $B^I, A, S > 0$) and (3.2.8a) is the consumption Euler equation, (3.2.8b) governs the trade-off between consumption in old age and the bequests that are left for the offspring and (3.2.8c) describes the trade-off between unintended and intended bequests.

The condition in (3.2.8c) immediately reveals that the trade-off between unintended and intended bequests – and, thereby, between bequests and old age consumption – depends critically on the difference between the interest rate on annuities and the rate on savings. To set ideas assume that annuities are priced at an actuarially fair rate. In that case:

$$1 + r^A(\mu) = \frac{1+r}{\mu}, \tag{3.2.9}$$

implying that purchasing annuities exactly offsets the life-time uncertainty faced by individuals. Substituting (3.2.9) into (3.2.8c) implies that $V'(B^U) = V'(B^I)$, which by virtue of the assumptions surrounding (3.2.7) implies $B^U = B^I$. Combining the latter with (3.2.2) and (3.2.3) provides $C^O = (1 + r^A(\mu))A$, which is the result of Abel (1986) and Davidoff *et al.* (2005) outlined in the introduction. That is, with actuarially fair annuities individuals annuitize the part of their assets that they wish to use consumption in old age. In that sense, bequest motives by themselves cannot explain the annuity puzzle as they still imply substantial degrees of annuitization.

If we abstract from bequest motives, only condition (3.2.8a) remains relevant for the individual optimization problem. In that case the trade-off induced by bequests between non-annutized savings and annuities disappears and individuals chose the financial asset with the highest return. This implies that unless annuities are priced extremely unfair (the annuity premium would have to be negative), individuals will always annuitize all their savings as the return that they receive on them is higher than the return on uninsured assets (see, Davidoff *et al.* (2005)). This leads to a counter-factual prediction indicating that mis-priced annuities in isolation can also not rationalize the observed low levels of annuitization.

In sum, in addition to outlining our analytical framework, in this section we have briefly analyzed two commonly considered reasons for non-annuitization – bequests and unfair pricing. For both cases we show that standard model cannot explain low levels of annuitization. This does not, however, mean that bequest motives and mispricing are irrelevant for the annuity puzzle. In fact, as we outline in the remainder of the paper, while individually they cannot rationalize puzzle, jointly they can.

3.3 Bequest Motives & Asymmetric Information

To analyze the role of asymmetric information concerning health states and bequest motives for non-annuitization, we start by focusing on the fact that individuals are heterogeneous with respect to their health types. With health types being private information, healthy individuals have an incentive to claim being unhealthy as this would increase the premium that they receive on their annuities (see Eq. (3.2.9))

3.3.1 Non-Annuitization

Relegating the impact of adverse selection to Section 3.3.2, we assume that the government sets up an non-mandatory annuity plan that offers a single – pooling – contract to all potential participants:

$$1 + r^A(\bar{\mu}) = \frac{1+r}{\bar{\mu}}, \tag{3.3.1}$$

where $\bar{\mu}$ is the average mortality rate. With (3.3.1) in hand, the individual budget constraint becomes:

$$C^O(\mu) + B^I(\mu) = (1 + r^A(\bar{\mu}))(w - C^Y(\mu)) - \frac{r^A(\bar{\mu}) - r}{1+r} B^A(\mu), \tag{3.3.2}$$

where we have included μ as an argument to choice variables to highlight their dependence on individual mortality.

As before, the objective of the individual is to maximize (3.2.5) subject to (3.3.2) and a set of borrowing constraints on bequests, annuities and savings. Consequently, assuming an interior solution for bequests and savings, the first order conditions become:

$$\begin{aligned}
U'(C^Y(\mu)) - (1 + r^A(\bar{\mu}))\frac{\mu}{1+\rho}U'(C^O(\mu)) &= 0, && \text{(3.3.3a)}\\
U'(C^O(\mu)) - V'(B^I(\mu)) &= 0, && \text{(3.3.3b)}\\
V'(B^U(\mu)) - \frac{r^A(\bar{\mu}) - r}{1+r}\frac{\mu}{1-\mu}V'(B^I(\mu)) &= 0, \ (= 0 \text{ if } A(\mu) > 0), && \text{(3.3.3c)}
\end{aligned}$$

where in the final line we take into account that depending on whether the non-negativity constraint on annuity holdings is binding or not, the first order condition differs slightly. Crucially, the possibility that individuals can rationally choose not to hold annuities whilst having positive savings is what sets our analysis apart from that of Abel (1986). Indeed, while he limits his analysis to individuals with positive annuity holdings, we let individuals decide how many, if any, annuities to hold.

To consider the possibility of non-annuitization further it serves to impose the functional forms laid out in (3.2.6) and (3.2.7). With those in hand, individual demand for annuities is given by:

$$A(\mu) = \left[1 + \eta^\sigma - \left[\frac{r^A(\bar{\mu}) - r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^\sigma\right]\phi(\mu, r^A(\bar{\mu}))\left[w + \frac{\theta}{1+r}\right], \tag{3.3.4}$$

where $\phi(\mu, r^A(\bar{\mu}))$ is given by:

$$\phi(\mu, r^A(\bar{\mu})) = \left[(1+r^A(\bar{\mu}))^{1-\sigma}\left(\frac{\mu}{1+\rho}\right)^{-\sigma} + \left(\frac{r^A(\bar{\mu})-r}{1+r}\right)^{1-\sigma}\left(\frac{\mu}{1-\mu}\right)^{-\sigma}\eta^{\sigma} + 1 + \eta^{\sigma}\right]^{-1}. \tag{3.3.5}$$

From (3.3.4) we can observe that annuity demand will be zero if:

$$1 + \eta^{\sigma} - \left[\frac{r^A(\bar{\mu})-r}{1+r}\frac{\bar{\mu}_C}{1-\bar{\mu}_C}\right]^{-\sigma}\eta^{\sigma} = 0, \tag{3.3.6}$$

where $\bar{\mu}_C$ is the cut-off health type. That is, individuals with a survival probability that is equal to or smaller than $\bar{\mu}_C$ will not purchase any annuities even though they hold positive savings. Intuitively, for these individuals the gain from annuitization (i.e., a higher premium) does not outweigh the costs of annuitization (i.e., the welfare loss of not leaving a bequest) because the premium that they can get is insufficient. Hence, the combination of bequest motives and under-priced annuities can justify why individuals do not hold annuities. Importantly, the fact that annuity holdings are lower for individuals with worse health is in line with empirical evidence provided by, for instance, Inkmann *et al.* (2007), who show that individuals holding annuities have a higher subjective survival probability (our μ) than those who do not.

3.3.2 Adverse Selection

To generate a better understanding of how non-annuitization by low-health types affects the annuity premium we now move from a government sponsored to a private annuity market. The main discrepancy between these two options is that in the latter case annuity firms take into account that offering the rate that the government offers would lead them to make losses. Therefore, the rate they offer will be based on the asset weighted average mortality rate:

$$1 + r^A(\mu^A) = \frac{1+r}{\mu^A}, \tag{3.3.7a}$$

$$\text{where} \quad \mu^A = \frac{\int_{\mu_C}^{\mu_H} \mu A(\mu) h(\mu) d\mu}{\int_{\mu_C}^{\mu_H} A(\mu) h(\mu) d\mu} \tag{3.3.7b}$$

is the asset weighted average survival probability and μ_C is the cut-off value of health types not wanting to annuitize when the impact of adverse selection on the annuity premium is taken into account. Repeating the steps leading up to (3.3.4), the non-annuitization condition of (3.3.6) now becomes:

$$1 + \eta^{\sigma} - \left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu_C}{1-\mu_C}\right]^{-\sigma}\eta^{\sigma} = 0. \tag{3.3.8}$$

So that the solution to μ_C can now be determined by solving (3.3.7) and (3.3.8) jointly. For the current purpose we are not, however, interested in the exact value of μ_C but in its relative value with respect to $\bar{\mu}_C$ – as described in the following proposition.

Proposition 1. Adverse selection increases the cut-off health type. That is, $\mu_C > \bar{\mu}_C$.

Proof. Equations (3.3.6) and (3.3.8) reveal that $\mu_C > \bar{\mu}_C$ if and only if $r^{AS} < \bar{r}$. From (3.3.1) and (3.3.7) this condition will be met if $\mu^A > \bar{\mu}$, which is known to be true from Abel (1986). □

Hence, while pooling in combination with bequest motives can account for a share of non-annuitization observed in financial markets, adverse selection aggravates non-annuitization further in the sense that ever healthier individuals rationally choose to forgo annuitizing their assets.

3.3.3 Numerical Example

To give more sense to the theoretical arguments outlined above we use this section to provide a brief numerical example of our model. To this end, let the interest rate and rate of time preferences both be 3%. Assuming that both periods last 40 years this implies $r = \rho = 2.26$. We obtain a consistent wage rate by imposing a Cobb-Douglas production function with unit total factor productivity which, assuming a capital share of output equal to 0.3, leads to $w = 13.18$. As concerns the utility function parameters we let the inter-temporal elasticity of substitution equal $\sigma = 1$, we assume that individuals derive less utility from bequests than from own consumption by setting $\eta = 0.5$ and we let the bequest threshold equal 20% of first period income. Finally, we assume that health types are distributed uniformly between survival probabilities $\mu_L = 0.1$ and $\mu_H = 0.9$.[6]

Using the various parameter values outlined above provides $\bar{\mu}_C = 0.25$ and $\mu_C = 0.54$. This implies that of all individuals who drop out of the annuity market about half drops out purely due to the existence of a pooling equilibrium as represented by a government sponsored annuity market, while the other half drops out due to the adverse selection induced by asymmetric information in, for instance, a privatized annuity market. Importantly, this emphasizes the point previewed in the introduction that the impact of asymmetric information on annuitization is a two-step process. First, a part of the population ceases to annuitize purely due to the existence of the pooling equilibrium caused by the government mandated annuity rate. Second, a further part quits the annuity market because of the adverse selection induced by the first stage makes the annuity premium ever more unfair for low health types.

Figure 4.1 visualizes the arguments outlined in this section. In particular it shows how the cut-off value for health types who still annuitize is affected by the pooling equilibrium and by adverse selection. Indeed, while in the absence of both all health

[6] Our results are robust to various other parameter constellation, available upon request.

Figure 3.1: Retreat from Annuity Market due to Pooling and Adverse Selection

Note. PE stands for Pooling Equilibrium. AS stands for adverse selection.

types annuitize that part of their savings that they wish to consume in old-age, the pooling equilibrium eliminates the most unhealthy types from the market. This effect is then further aggravated through adverse selection which causes ever healthier individuals to retreat from the annuity market.

3.4 Further Analysis

Having established how the combination of bequest motives and asymmetric information may lead to rational non-annuitization, we use this section to explore the implications of *pay-as-you-go* social security and the health-wealth nexus. Earlier analyses have suggested that social security might be a reason for why annuity holdings among individuals is limited and we use this section to assess how it affects the cut-off health value below which individuals do not annuitize savings. Similarly, we assess how the cut-off value is affected by a correlation between earnings and health.

3.4.1 Social Security

A common rationalization for low observed levels of annuitization is that individuals have covered their longevity risk through pre-existing mandatory public pension arrangements such as social security. However, while social security may crowd out savings in general, there is no reason why its existence would lead individuals to not annuitize their private savings in the absence of additional market imperfections. In this subsection we develop this point further by considering the impact of a pay-as-you-go social security system on our above results.

The social security system is financed by imposing a tax on the young, which is redistributed to the currently old in the form of a pension payment. Formally, the system implies that, when young, individuals pay a tax T and in return, conditional on survival, the pension payment equals $(1 + r^S)T$, where r^S is the internal rate of return of the social security system. Letting T be constant we can determine r^S by employing the budget constraint of the social security system:

$$\int_{\mu_L}^{\mu_H} Th(\mu)d\mu = \int_{\mu_L}^{\mu_H} (1 + r^S)T\mu h(\mu)d\mu, \tag{3.4.1}$$

where the left-hand-side is the income of the social security system while the right-hand-side are its expenditures. Solving (3.4.1) for r^S provides:

$$1 + r^S = \frac{1}{\bar{\mu}}, \tag{3.4.2}$$

which may be higher or lower than the rate received in the adverse selection equilibrium $r^A(\mu^A)$, derived above.[7]

Taking into account the social security system, the life-time budget constraint now becomes:

$$C_2(\mu) + B^I(\mu) = (1 + r^A(\mu^A))(w - C_1(\mu)) - \frac{r^A(\mu^A) - r}{1 + r} B^A(\mu) + (r^S - r^A(\mu^A))T. \tag{3.4.3}$$

Comparing the budget constraint in (3.4.3) to the previous one in (3.3.2), the augmented term on the right-hand-side reflects the increased (decreased) life-time resources due to higher (lower) return from the social security system.

Repeating the steps outlined in the sections above, the optimal amount of annuity purchases is as follows:

$$A(\mu) = \left[1 + \eta^\sigma - \left[\frac{r^A(\mu^A) - r}{1 + r} \frac{\mu}{1 - \mu}\right]^{-\sigma} \eta^\sigma\right] \phi(\mu, r^A(\mu^A)) \left[w + \frac{r^S - r^A(\mu^A)}{1 + r^A(\mu^A)} T + \frac{\theta}{1 + r}\right] - \frac{1 + r^S}{1 + r^A(\mu^A)} T. \tag{3.4.4}$$

Defining μ_C^{SS} as the cut-off health value for which individuals no longer annuitize (i.e., $A(\mu_C^{SS}) = 0$), we observe that as long as $T > 0$, μ_C must satisfy:

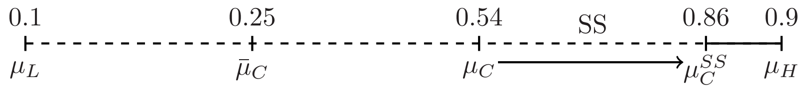
$$1 + \eta^\sigma - \left[\frac{r^A(\mu^A) - r}{1 + r} \frac{\mu_C^{SS}}{1 - \mu_C^{SS}}\right]^{-\sigma} \eta^\sigma > 0. \tag{3.4.5}$$

Comparing (3.3.8) to (3.4.5) immediately reveals that for given $r^A(\mu^A)$, the cut-off health value μ_C^{SS} increases after the introduction of the social security system. Naturally, as μ_C^{SS} increases, $r^A(\mu^A)$ decreases which causes ever more individuals to abandon the annuity market. Intuitively, this occurs because the social security system crowds out savings by low-health types so that they drop out of the financial market altogether, thereby also aggravating adverse selection in the annuity market.

In Figure (3.2) we visualize the above effect once more by alluding to the numerical example introduced above. For sake of argument we let the social security contribution be a 20% of the wage rate implying that $T = 0.2w$ which provides a social security return of $r^S = 1$, implying a benefit rate of 0.4. As the figure highlights, the social security system further aggravates adverse selection in the annuity market. Indeed, upon the imposition of the system, only very healthy individuals remain in the annuity market. In particular, only those with a survival probability of more than 86%.

[7] Alternatively, and equivalently, we can define the social security system in terms of a tax rate τ^{SS} and a benefit rate β^{SS}, which are both proportional to the wage rate. Condition (3.4.3) then becomes the familiar expression: $\tau^{SS}/\beta^{SS} = 1/\bar{\mu}$.

Figure 3.2: Non-annuitization and Social Security



Note. SS stands for Pay-As-You-Go Social Security.

This is not, however, the complete picture as even the very healthy types adjust their savings behavior due to the presence of the social security system. In particular, they reduce their total savings and, more importantly, they reduce the share of savings held in the form of annuities significantly. This result arises because social security is non-bequeathable so that in the presence of bequest motives, social security leads to a relatively stronger crowding out of annuities than of non-annuitized savings. We visualize this portfolio re-balancing in Table 3.1. As can be seen in the table, for the most healthy individuals, total savings $(A + S)$ decline but the total amount of non-annuitized savings actually increases. Hence, the crowding out of savings through the social security system is captured by savings held in annuitzed accounts. We formalize this intuition in the following proposition.

Proposition 2. A pay-as-you-go social security system leads to a re-balancing of retirement portfolios toward non-annuitized assets.
Proof. In the presence of social security the ratio of annuities to non-annuitized savings in the retirement portfolio is given by:

$$\frac{A^{SS}(\mu)}{S^{SS}(\mu)} = \frac{\left[1 + \eta^{\sigma} - \left[\frac{r^A(\mu^A) - r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}\right] - \gamma\frac{1+r^S}{1+r^A(\mu^A)}T}{\left[\frac{r^A(\mu^A) - r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}}.$$

To show that an increase in the size of the social security system reduces this ratio consider that the sign of $\frac{d\frac{A^{SS}(\mu)}{S^{SS}(\mu)}}{dT}$ equals the sign of $\frac{d\ln\left(\frac{A^{SS}(\mu)}{S^{SS}(\mu)}\right)}{dT}$:

$$\frac{d\ln\left(\frac{A^{SS}(\mu)}{S^{SS}(\mu)}\right)}{dT} = \frac{1}{\left[1+\eta^{\sigma}-\left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}\right]-\gamma\frac{1+r^S}{1+r^A(\mu^A)}T}$$
$$\left(-\frac{d\left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}}{dT}-\gamma\frac{1+r^S}{1+r^A(\mu^A)}-\gamma\frac{1+r^S}{(1+r^A(\mu^A))^2}T\left(-\frac{dr^A(\mu^A)}{dT}\right)\right)$$
$$-\frac{1}{\left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}}\frac{d\left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}}{dT},$$

Since $\frac{dr^A(\mu^A)}{dT}<0$, we have $\frac{d\left[\frac{r^A(\mu^A)-r}{1+r}\frac{\mu}{1-\mu}\right]^{-\sigma}\eta^{\sigma}}{dT}>0$ and therefore: $\frac{d\ln\left(\frac{A^{SS}(\mu)}{S^{SS}(\mu)}\right)}{dT}<0$. Hence, an increase in the size of the social security system decreases the relative amount of assets held in the form of annuities.

□

Table 3.1: Numerical Example

	AS			SS		
	μ_C	$\bar{\mu}^{AS}$	μ_H	μ_C	$\bar{\mu}^{SS}$	μ_H
C^Y	0.2302	0.2209	0.2124	0.2064	0.2056	0.2048
C^O	0.1594	0.2043	0.2458	0.2010	0.2043	0.2075
B^A	0.2096	0.1101	0.0183	0.0953	0.0787	0.0622
B^I	0.0502	0.0727	0.0934	0.0710	0.0727	0.0743
A	0.0000	0.0398	0.0765	0.0000	0.0059	0.0117
S	0.0643	0.0338	0.0056	0.0292	0.0241	0.0191

Note: Benchmark case. $T = 0.2w$. $\bar{\mu} = (\mu_C + \mu_H)/2$.

3.4.2 Health-Wealth Nexus

Whilst the preceding analysis has taken into account heterogeneity between individuals in terms of their survival probabilities, we use this subsection to extend the degree of heterogeneity to a correlation between earnings ability and health. To this end, we take inspiration from, amongst others, Currie (2007), who convincingly shows that health and wealth are correlated due to – at least partially – a positive relationship between health and earnings ability. We implement this relationship in our model by letting an individual with better health have a higher first period income, $w'(\mu) > 0$. This implies that a type–μ individual's intertemporal budget constraint becomes:

$$C^O(\mu) + B^I(\mu) = \left(1 + r^A(\mu^{HW})\right)\left(w(\mu) - C^Y(\mu)\right) - \frac{r^A(\mu^{HW}) - r}{1+r} B^A(\mu), \quad (3.4.6)$$

where $r^A(\mu^{HW})$ is the rate on annuities when we take into account the correlation between earnings and health:

$$r^A(\mu^{HW}) = \frac{1+r}{\mu^{HW}}, \quad (3.4.7a)$$

$$\text{where} \quad \mu^{HW} = \frac{\int_{\mu_C}^{\mu_H} \mu A^{HW}(\mu) h(\mu) d\mu}{\int_{\mu_C}^{\mu_H} A^{HW}(\mu) h(\mu) d\mu}. \quad (3.4.7b)$$

As before we can determine annuity purchases using the steps outlined above, this provides:

$$A^{HW}(\mu) = \left[1 + \eta^\sigma - \left[\frac{r^A(\mu^{HW}) - r}{1+r} \frac{\mu}{1-\mu}\right]^{-\sigma} \eta^\sigma\right] \phi(\mu, r^A(\mu^{HW})) \left[w(\mu) + \frac{\theta}{1+r}\right]. \quad (3.4.8)$$

Viewing (3.4.8) immediately implies that the cut-off value for health types is implicitly determined by:

$$1 + \eta^\sigma - \left[\frac{r^{HW} - r}{1+r} \frac{\mu_C^{HW}}{1-\mu_C^{HW}}\right]^{-\sigma} \eta^\sigma = 0, \quad (3.4.9)$$

in combination with (3.4.7). This brings us to the following proposition.

Proposition 3. The health-wealth nexus aggravates adverse selection in the annuity market:

Proof. Since $w'(\mu) > 0$, $A^{HW}(\mu) - A(\mu)$ is positively related to μ. First we show that $r^A(\mu^{HW}) < r^A(\mu^A)$, which is equivalent in showing that $\mu^{HW} > \mu^A$. Using the definitions of μ^A and μ^{HW} from (3.3.7) and (3.4.7), respectively, we can write:

$$\mu^{HW} - \mu^A = \frac{\int_{\mu_C}^{\mu_H} \left[A^{HW}(\mu) - A(\mu)\right](\mu - \mu^A) h(\mu) d\mu}{\int_{\mu_C}^{\mu_H} A^{HW}(\mu) h(\mu) d\mu} \quad (3.4.10)$$

$$= \frac{cov\left(\left[A^{HW}(\mu) - A(\mu)\right], \mu\right)}{\int_{\mu_C}^{\mu_H} A^{HW}(\mu) h(\mu) d\mu} > 0, \quad (3.4.11)$$

so that: $r^A(\mu^{HW}) < r^A(\mu^A)$. By comparing (3.3.8) to (3.4.9) this automatically implies $\mu_C^{HW} > \mu_C$. □

In words, accounting for the health-wealth nexus aggravates adverse selection in the private annuity market. Intuitively, this result arises from the fact that the health-wealth nexus increases the amount of annuities held by healthy individuals so that the annuity premium drops, leading more lower health types to abandon the annuity market.

3.4.3 An Afterthough on Administration Costs

We now close the analysis by analyzing the impact of a mark-up due to, for instance, administration costs but also monopoly profits or marketing costs on the extent of adverse selection in the annuity market. These mark-ups were initially popularized by Mitchell *et al.* (1999) to calculate the money's worth of annuities who attribute them to administration costs as well as adverse selection.[8] We have, however, already addressed adverse selection above.

To this end let the return on annuities equal:

$$1 + r^A(\lambda, \mu) = (1 - \lambda)(1 + r^A(\mu^A)), \tag{3.4.12}$$

where λ is a so-called load factor, which measures the percentage reduction in the annuity premium due to administration costs. The higher λ, the higher the administration costs. Keeping λ in mind, the life-time budget constraint becomes:

$$C_2(\mu)+B^I(\mu) = \left(1 + r^A(\lambda, \mu)\right)(w - C_1(\mu)) - \frac{r^A(\lambda, \mu) - r}{1 + r} B^U(\mu) + \left(r^S - r^A(\lambda, \mu)\right) T. \tag{3.4.13}$$

Without testing the patience of the reader, we can use the new budget constraint in (3.4.12) to derive the new annuity demand. Doing so and setting the ensuing value of $A(\mu_C^{AC}) = 0$ provides the new cut-off value of health type who do not annuitize, μ_C^{AC}. By observing (3.4.12) we immediately note that $\mu_C^{AC} > \mu_C$ so that the administration costs aggravate adverse selection in the annuity market. By similar reasoning as above we see that this follows from the fact that administration costs make annuities unattractive in general and in particular for the unhealthier types. Moreover, once the administration costs induce low health types to retreat from the annuity market, the annuity premium drops further hence inducing ever more individuals to abandon the annuity market. Importantly, this provides a distinction between our analysis and that of Lockwood (2012) and the early analysis of Mitchell *et al.* (1999) in the sense that our current result shows the various components generally attributed to the annuity load factor amplify each other.[9]

3.5 Conclusion

In this Chapter we analyzed the interplay between asymmetric information concerning individual health and bequest motives for the decision of individuals to annuitize their retirement savings. While asymmetric information and bequest motives in isolation cannot account for why individuals do not annuitize, we show that their interplay can. In particular, because bequest motives give additional value to

[8] Such mark-ups are a common feature in the analysis of annuities (see, for instance, Bütler (2001), Hansen and İmrohoroğlu (2008), Heijdra and Mierau (2012) and Lockwood (2012)).

[9] It goes without saying that for sufficiently large administration costs even the highest health types would not annuitize anymore. Similarly, a sufficiently large social security system would also entice the healthiest types to leave the annuity market.

non-annuitized savings, under-priced annuities due to asymmetric information are very unattractive for low health individuals. Analyzing our result further we show that the presence of a social security system further aggravates adverse selection and, in addition, leads healthier individuals to re-balance their retirement savings portfolio away from annuities. The latter being a consequence of the fact that both annuities and social security are non-bequeathable. Finally, taking into account that health and wealth are potentially correlated we show that higher wealth holdings by healthier individuals drive ever more unhealthy individuals out of the annuity market.

Chapter 4

Annuities and Bequests in General Equilibrium*

*This chapter is based on Heijdra, Jiang and Mierau (2017b).

4.1 Introduction

Bequests motives are often invoked to justify the discrepancy between the theoretical benefits of annuities displayed in Yaari (1965) and their low uptake as highlighted by, amongst many others, Inkmann *et al.* (2011). Indeed, Lockwood (2014) shows that the presence of bequest motives significantly decreases the enthusiasm for annuities because they assign a utility value to assets that in the presence of annuities would have gone into the pool of annuity assets. The previous Chapter reinforces this point by showing that if asymmetric information, administration costs or social security are considered, the demand for annuities may very well vanish. Yet in spite of the above, in many situations a positive demand for annuities prevails – at least theoretically. After all, not withstanding bequest motives and the like, actuarially fair annuities – or at least not too unfair – provide a higher return than otherwise equivalent but not life-insured assets (Davidoff *et al.*, 2005).

Private welfare gains of annuities need not, however, translate into public virtues. Substantiating this point, a literature is emerging that juxtaposes the partial and general equilibrium benefits of annuitization. Pecchenino and Pollard (1997), Fehr and Habermann (2008), Feigenbaum *et al.* (2013) and Caliendo *et al.* (2014) point toward the observation that private and public benefits of annuities differ due to the loss of unintended bequests. Christening the ensuing problem as the *Tragedy of Annuitization*, Heijdra *et al.* (2014) show that the welfare effects can best be understood by reference to the Golden Rule. In particular, in the absence of annuities any unconsumed assets flow from older to younger generations. Considering a dynamically efficient economy, this intergenerational transfer constitutes a reverse social security system, which drives the economy closer to the Golden Rule steady state. Introducing annuities cuts off this intergenerational flow of assets and, therefore, reduces capital accumulation and welfare.

Whilst the extant literature concerning the discrepancy between private and public benefits of annuities have centered on *unintended* bequests, the role of *intended* bequests has received only muted attention. However, by attaching a utility value to bequests – as highlighted in the partial equilibrium literature outline above – bequests motives can potentially mitigate the general equilibrium welfare loss stemming from the opening of an annuity market. Arguing along these lines, Heijdra *et al.* (2017b), for instance, provide an informal discussion of the potential role of intentional bequests for the general equilibrium welfare consequences of annuitization.

Against the above backdrop we use the current Chapter to further elucidate the general equilibrium consequences of opening up an annuity market in the presence of intentional bequest motives. To this end, we develop a tractable two-period overlapping generations model in which individuals face mortality risk in the transition from the first period to the next. We operationalize the bequest motive along the lines of the partial equilibrium models of Abel (1986), Lockwood (2012) and Heijdra *et al.* (2017a), who consider bequests as gifts from which individuals derive utility. In keeping with Abel (1986), we embed the individual life-cycle model in a general equilibrium context with perfectly competitive firms. In contrast to Abel, however, we do not consider the role of social security but focus on the impact of opening up an annuity market on capital accumulation and individual welfare.

Within our framework we set out by focusing on the role of bequest motives in general equilibrium. Here we show that stronger bequest motives are associated with higher capital accumulation because they entail a redistribution of assets from the older to the younger generation. We then proceed by analyzing the impact of opening up an annuity market on capital accumulation and welfare. Here we derive our main result that, while the welfare loss of annuities is more muted in the presence of bequest motives, the *Tragedy of Annuitization* also prevails when bequest motives are accounted for. Indeed, if it is individually optimal to annuitize at least a part of assets, then opening up an annuity market will decrease the flow of assets from older to younger generations, thereby decreasing capital accumulation and, hence, welfare.

The remainder of the paper is set-up as follows. Section 4.2 sets out the model in the absence of annuities and focuses on the role of bequests motives on capital accumulation. In Section 4.3 we add annuities to the model and assess the impact of opening up an annuity market on capital accumulation and welfare. Section 4.4 provides some further analysis and the final section concludes. The Appendix collects the proofs of our propositions.

4.2 Bequests in General Equilibrium

Our general set-up is derived from Abel (1986) and follows the analysis in Chapter 3 . In contrast to our earlier partial equilibrium analysis we disregard heterogeneity in survival probabilities but instead embed the individual life-cycle model in a general equilibrium context. In this way we can focus on the general equilibrium impact of annuities and bequests in a tractable model.

4.2.1 Individuals

Individuals live for a maximum of two periods - youth and old age. They work full time in the first period and are retired in the second. Survival from the first to the second period is governed by a probability μ. If an individual dies at the end of the first period, his/her heir receives an unintended bequest B^U. If an individual survives to the second period, (s)he divides wealth between old age consumption and intentional bequests. Following Abel (1986), we assume that survivors will leave bequests B^I at the beginning of the second period. Hence, regardless of whether the individual lives one or two periods, the intergenerational transfer always takes place at beginning of the second period.[1]

To set ideas, we first consider the benchmark case in which individuals have no access to annuities. Expected life-time utility of an individual born at time t is then

[1] In that sense, we may also refer to the intentional bequests as inter-vivos transfers.

given by:

$$\mathbb{E}\Lambda_t = U(C_t^Y) + \frac{1-\mu}{1+\rho} V(B_{t+1}^U) + \frac{\mu}{1+\rho} \left[U(C_{t+1}^O) + V(B_{t+1}^I) \right], \tag{4.2.1}$$

where ρ is the pure rate of time preference and C_t^Y and C_{t+1}^O are consumption in youth and old age, respectively.

$U(C)$ and $V(B)$ are increasing and concave utility functions of consumption and bequests, respectively. In keeping with the extant literature, we parameterize both as iso-elastic functions with an intertemporal elasticity of substitution equal to $\sigma \in (0, 1)$. The consumption-utility function then becomes:

$$U(C) = \begin{cases} \frac{C^{1-1/\sigma}-1}{1-1/\sigma} & \text{if} \quad \sigma > 0, \quad \sigma \neq 1, \\ \ln C & \text{if} \quad \sigma = 1. \end{cases} \tag{4.2.2}$$

Similarly, the bequest-utility function is:

$$V(B) = \begin{cases} \eta \frac{(\theta+B)^{1-1/\sigma}-1}{1-1/\sigma} & \text{if} \quad \sigma > 0, \quad \sigma \neq 1, \\ \eta \ln(\theta + B) & \text{if} \quad \sigma = 1, \end{cases} \tag{4.2.3}$$

where we have followed the common practice of letting the intertemporal elasticity of substitution be equal across consumption and bequests. In addition, $\eta \geq 0$ describes the strength of the bequest motive and $\theta \geq 0$ is the threshold wealth below which individuals do not leave bequests. The structure in (4.2.3) is sufficiently general to reconcile commonly used functional forms (see, Pashchenko (2013) for a review).

Following Abel (1986), Hong and Rios-Rull (2007), Caliendo *et al.* (2014) and others we assume that accidental and intentional bequests of deceased individuals are collected by the government and distributed to the next generation. The first period budget constraint can then be written as:

$$C_t^Y + S_t = w_t + Z_t, \tag{4.2.4}$$

where w_t is wage income, S_t are savings and Z_t are the distributed bequests. If an individual dies, all savings (including their accrued return) are left behind as unintended bequests implying that:

$$B_{t+1}^U = (1 + r_{t+1}) S_t, \tag{4.2.5}$$

where r_{t+1} is the interest rate. If an individual survives, (s)he can distribute the retirement savings between consumption goods and intentional bequests:

$$C_{t+1}^O + B_{t+1}^I = (1 + r_{t+1}) S_t. \tag{4.2.6}$$

Consolidating the periodic budget constraints in (4.2.4) and (4.2.6) yields:

$$C_t^Y + \frac{C_{t+1}^O}{1 + r_{t+1}} + \frac{B_{t+1}^I}{1 + r_{t+1}} = w_t + Z_t. \tag{4.2.7}$$

The objective of the consumer is to maximize discounted life-time utility provided in (4.2.1) subject to the consolidated budget constraint provided in (4.2.7) by choice of C_t^Y, C_{t+1}^O and B_{t+1}^I. The ensuing consumption, savings and bequest plans are given by:

$$C_t^Y = (1-\phi(r_{t+1}))\left[w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right], \tag{4.2.8a}$$

$$C_{t+1}^O = \frac{1}{1+\eta^\sigma}(1+r_{t+1})\phi(r_{t+1})\left[w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right], \tag{4.2.8b}$$

$$S_t = \phi(r_{t+1})\left(w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right) - \frac{\theta}{1+r_{t+1}}, \tag{4.2.8c}$$

$$B_{t+1}^I = \frac{\eta^\sigma}{1+\eta^\sigma}(1+r_{t+1})\phi(r_{t+1})\left[w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right] - \theta, \tag{4.2.8d}$$

$$B_{t+1}^U = (1+r_{t+1})\phi(r_{t+1})\left[w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right] - \theta, \tag{4.2.8e}$$

where $\phi(r_{t+1}) \in (0,1)$ is the marginal propensity to save out of total wealth during youth:

$$\phi(r_{t+1}) = \left[\frac{1+r_{t+1}}{1+\eta^\sigma}\left(\frac{1+r_{t+1}}{1+\rho}\left[\left(\frac{\eta^\sigma}{1+\eta^\sigma}\right)^{1/\sigma}(1-\mu)+\mu\right]\right)^{-\sigma} + 1\right]^{-1}. \tag{4.2.9}$$

The expression in (4.2.8c) allows us to consider the partial equilibrium relationship between bequest motives and individual capital accumulation (i.e., savings). Starting with the strength of the bequest motive, η, we observe that $\frac{\partial \phi}{\partial \eta} > 0$. Intuitively, consumers with stronger bequest motives are more inclined to save. Therefore, a stronger bequest motive decreases youth consumption, C_t^Y, and increases savings, S_t as well as unintended and intended bequests, B_{t+1}^I and B_{t+1}^U, respectively. Conversely, a stronger threshold wealth, θ, makes the marginal bequest less attractive and, therefore, decreases savings. Indeed, straightforward differentiation of (4.2.8c) with respect to θ provides $\frac{\partial S_t}{\partial \theta} < 0$. While these are partial equilibrium results, we use the following section to highlight that individual results concerning capital accumulation and bequests translate into a positive aggregate relationship between the strength of the bequest motive and capital accumulation. Similarly, a higher threshold wealth decrease aggregate capital accumulation.

4.2.2 Aggregate Economy

Demography

Let the size of the young cohort at time t be L_t and the population growth rate be n then in every new period a cohort of size $L_{t-1} = (1+n)L_t$ is born. Aggregating over both cohorts implies that the total population in the economy at time t is given by:

$$P_t = \mu L_{t-1} + L_t = \frac{1+n+\mu}{1+n}L_t, \tag{4.2.10}$$

where we have taken into account that due to lifetime uncertainty only a fraction μ of the young cohort survive into the next generation.

Production

There is a large number of perfectly competitive firms producing output according to a constant-returns to scale production function of the Cobb-Douglas form:

$$Y_t = \Omega K_t^{\varepsilon} L_t^{1-\varepsilon}, \quad 0 < \varepsilon < 1, \tag{4.2.11}$$

where Y_t is total output, Ω is total factor productivity, K_t is the aggregate capital stock and L_t is the size of the young cohort (i.e., the labor force). In intensive form, the production function can be written as:

$$y_t = \Omega k_t^{\varepsilon}, \tag{4.2.12}$$

where $y_t \equiv Y_t/L_t$ is output per capita and $k_t \equiv K_t/L_t$ is capital intensity. Assuming perfect competition, factor prices equal:

$$r_t = \varepsilon\Omega_0 k_t^{\varepsilon-1} - \delta \tag{4.2.13a}$$

$$w_t = (1-\varepsilon)\Omega_0 k_t^{\varepsilon}, \tag{4.2.13b}$$

where δ is the depreciation rate.

Government

As outlined above, following earlier contributions, we assume that the government collects intended and unintended bequests and redistributes them to the next generation. Total bequests in any given period equal $((1-\mu)B_t^U + \mu B_t^I)L_{t-1}$, these are distributed to all young cohort members in the form of a bequest transfer Z_t, implying the following government budget constraint:

$$L_t Z_t = \left[(1-\mu)B_t^U + \mu B_t^S\right] L_{t-1} \Leftrightarrow Z_t = \frac{(1-\mu)B_t^U + \mu B_t^I}{1+n}. \tag{4.2.14}$$

Substituting (4.2.8d) and (4.2.8e) into (4.2.14) allows us to write the transfer as:

$$Z_t = \frac{(1-\mu)B_t^U + \mu B_t^I}{1+n} = \alpha(1+r_t)k_t + \frac{(\alpha-1)\theta}{1+n}, \tag{4.2.15}$$

where $\alpha \equiv (1+\mu) + \mu\frac{\eta^{\sigma}}{1+\eta^{\sigma}} \in (0,1)$. From the discussion below (4.2.9) we recall that B_t^U and B_t^I increase with the strength of bequest motive and decrease with the threshold wealth. Hence, from (4.2.15) it is clear that, keeping factor prices fixed, a stronger bequest motive increases the bequest transfer, while a higher threshold wealth decreases it. This can be verified by straightforward differentiation of (4.2.15).

4.2.3 Equilibrium

In equilibrium the capital market must clear, which implies that total savings of the young cohort equals the total capital stock: $K_{t+1} = L_t S_t$ or, in per worker terms, $k_{t+1} = S_t/(1+n)$. Using (4.2.8c) provides the fundamental difference equation that governs the accumulation of capital:

$$\begin{aligned} k_{t+1} &= \frac{S_t}{1+n} \\ &= \frac{1}{1+n}\left[\phi(r_{t+1})\left(w_t + Z_t + \frac{\theta}{1+r_{t+1}}\right) - \frac{\theta}{1+r_{t+1}}\right]. \end{aligned} \tag{4.2.16}$$

Proposition 1. There exists a unique, stable and non-trivial steady state at $k(t) = k$.

Proof. See Appendix A. ■

Importantly, with the steady-state solution in hand we can reassess the positive relationship between bequest motives and capital accumulation derived above using the following proposition:

Proposition 2. Capital intensity increases with the strength of the bequest motives and decreases with threshold wealth:

$$\frac{\partial k}{\partial \eta} > 0 \quad \text{and} \quad \frac{\partial k}{\partial \theta} < 0.$$

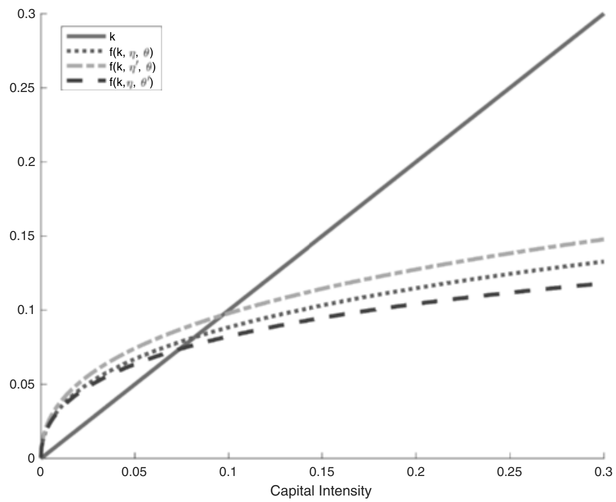
Proof. See Appendix B. □

Intuitively allowing for bequest motives will introduce an asset flow from older to younger generations. With younger generations using these assets for savings we observe a surge in savings (as pointed to above) and, consequently, an increase in aggregate capital accumulation. If, however, individuals have a high threshold wealth, the intergenerational flow of assets will be more muted so that capital accumulation will be retarded at the individual as well as the aggregate level. While tempting, we refrain from making statements about the impact of bequest motives on welfare as they change the structure of the utility function so that a clean comparison of welfare with and without bequest motives is not feasible.

4.2.4 Numerical Example

To provide more insights to the theoretical arguments developed above, we perform a numerical simulation of our model in the current section. To this end we associate numerical values to each of the model parameters. The annual capital depreciation rate - where we denote annualized rates with a hat - is six percent ($\hat{\delta} = 0.06$), the annual population growth rate is one percent ($\hat{n} = 0.01$), the time preference

Figure 4.1: Model Equilibrium



Note: The figure traces out the model equilibrium for various values of the bequest-motive parameters: $\eta = 0.5$ and $\theta = 0.10$ are the benchmark values, $\eta' = 0.7$ and $\theta' = 0.15$.

parameter is three percent ($\rho = 0.03$), the efficiency parameter of capital is thirty percent ($\varepsilon = 0.30$) and total factor productivity Ω is set to be 1. The inter-temporal elasticity of substitution is $\sigma = 0.7$. The benchmark strength of the bequest motive is chosen such that individuals derive half of the utility from leaving bequests compared to their own consumption, $\eta = 0.5$ and we scale θ such that it equals 10 % of labor income. We assume that consumers live for a maximum of two periods. The length of each period is set to be 40 years.[2] Finally, consumers face a survival probability of thirty percent at the end of the first period ($\mu = 0.3$).

In Figure 4.1 we trace out the left and right-hand side of (4.2.16). The finely dashed line in the middle is the benchmark equilibrium and it highlights the existence and uniqueness of the equilibrium. The upper line traces out the equilibrium with a stronger bequest motive and it shows that the capital stock is indeed higher when bequest motives are stronger. The lower line shows the equilibrium of the model when the threshold wealth necessary to leave bequest is high, which is associated to lower capital accumulation for reasons outlined above.

In Table 4.1 we display the various individual and aggregate quantities in equilibrium for different combinations of θ and η. The table shows the equilibrium values with annuities (labeled as A) and without annuities (NA); for now we focus on the former, in the next section we focus on the latter. As already highlighted in Proposition 2 as well as in Figure 4.1, stronger bequest motives are associated with higher

[2]The value of δ, n and r follow as $\delta = 1 - (1 - \hat{\delta})^{40}$, $n = (1 + \hat{n})^{40} - 1$, and $r = (1 + \hat{r})^{40} - 1$.

Table 4.1: Steady-State Values of Selected Model Variables ($\sigma = 0.7$)

	$\theta = 0.10$, $\eta = 0.5$		$\theta = 0.10$, $\eta = 0.7$		$\theta = 0.10$, $\eta = 1.0$		$\theta = 0.15$, $\eta = 0.5$	
	NA	A	NA	A	NA	A	NA	A
k	0.0817	0.0730	0.0967	0.0886	0.1166	0.1093	0.0737	0.0653
Y	0.4718	0.4561	0.4961	0.4833	0.5248	0.5148	0.4574	0.4411
Z	0.0745	0.0555	0.0860	0.0695	0.1001	0.0863	0.0673	0.0479
w	0.3302	0.3193	0.3473	0.3383	0.3673	0.3604	0.3202	0.3088
$\hat{r}$	0.0150	0.0169	0.0122	0.0137	0.0091	0.0101	0.0168	0.0188
C^Y	0.2830	0.2660	0.2894	0.2760	0.2939	0.2839	0.2777	0.2594
C^O	0.1572	0.1861	0.1509	0.1763	0.1429	0.1645	0.1619	0.1912
B^U	0.2210	0.0826	0.2337	0.1035	0.2490	0.1285	0.2135	0.0714
B^I	0.0637	0.0826	0.0828	0.1035	0.1061	0.1285	0.0516	0.0714
$\frac{A}{S+A}$	0.0000	0.6119	0.0000	0.5438	0.0000	0.4727	0.0000	0.6522
Λ	-2.7968	-2.8586	-2.9025	-2.9425	-3.0672	-3.0888	-2.8048	-2.8806

Note: NA and A stand for the case without and with annuity market, respectively.

capital accumulation while a higher threshold wealth is associated with lower capital accumulation. Having established the role of bequests for individual and aggregate capital accumulation, we proceed in the next section to study the role of opening up an annuity market on individual welfare in a general equilibrium setting. Importantly, this allows us to consider the impact of bequest motives on the discrepancy between individual and aggregate benefits of annuitization.

4.3 Annuities and Bequests in General Equilibrium

Annuities are life-insured financial assets that pay out conditional on the survival of the annuitant. If, in our case, the individual survives from the first period to the second, (s)he receives a return on his/her pension savings in excess of the market rate on uninsured savings. Conversely, if (s)he passes away, any annuitized assets flow to the annuity firm. From the seminal work of Yaari (1965) we know that in the absence of bequest motives it is individually optimal to annuitize all assets. In recent work, Lockwood (2012) and Chapter 3 show that bequest motives reduce the benefits of savings as they attach utility value to unconsumed assets.[3] Nevertheless, both, and others, show that in the presence of bequest motives it remains individually optimal to annuitize at least part the assets because it improves welfare. In this section we consider whether the increase in welfare also holds after taking into account general equilibrium repercussions of annuitization.

[3]Heijdra *et al.* (2017a) also consider the role of imperfect information, social security and administrative costs, all of which we abstract from in the current analysis to focus on the relationship between partial and general equilibrium benefits of annuitization.

4.3.1 Annuities

In the presence of annuity markets, the contributions from annuitants are pooled and invested in the capital market and the returns are then redistributed to the survivors. Hence, assuming that annuity firms break even, we have the following no-arbitrage condition:

$$(1 + r_{t+1})L_t A_t = (1 + r^A_{t+1})\mu L_t A_t, \tag{4.3.1}$$

with r^A being the return on annuities and where the left hand side are the total annuity holdings, A, of the young generation from time t and the right hand side is the pay out to the old generation in time $t + 1$ (*i.e.,* the surviving share of the young generation from the previous period). Obvious rewriting yields that the return of annuity equals the return on uninsured assets multiplied by a premium that is proportional to the probability of survival:

$$1 + r^A_{t+1} = \frac{1 + r_{t+1}}{\mu}, \quad 0 < \mu < 1, \tag{4.3.2}$$

which immediately yields that annuities are potentially attractive because they offer a higher rate of return than uninsured savings.

Should individuals annuitize (part of) their wealth, the temporal budget constraints become:

$$C^Y_t + S_t + A_t = w_t + Z_t, \tag{4.3.3a}$$

$$C^O_{t+1} + B^I_{t+1} = (1 + r_{t+1})S_t + (1 + r^A_{t+1})A_t. \tag{4.3.3b}$$

If consumers survive, their second period's wealth consists of two parts, the savings in the capital market and the investment in the annuity market:

$$Q_{t+1} = (1 + r_{t+1})S_t + (1 + r^A_{t+1})A_t, \tag{4.3.4}$$

where Q_{t+1} is the total wealth portfolio of an old individual. Obviously, purchasing annuities crowds out unintended bequests:

$$B^U_{t+1} = (1 + r_{t+1})S_t = W_{t+1} - (1 + r^A_{t+1})A_t. \tag{4.3.5}$$

In sum, purchasing annuities has the marginal benefit of a higher rate of return on assets and the marginal cost of lower accidental bequests.

In a companion paper, Chapter 3, we flesh out the relationship between bequests and individual savings in detail by also focusing on the role of differential mortality, administration costs and social security. The take-away message from that analysis is that the low uptake of annuities can be justified by the combination of bequest motives with various other common market features of the annuity market - such as asymmetric information about individual mortality and the existence of a social security system. For the current purpose we infer from our earlier paper that in spite of various market features, individuals still derive utility from having access to an annuity market. In what follows we focus in particular on whether the individual virtue of annuities translates into general equilibrium benefits as well.

With the discounted life-time utility function unchanged, the consumer's optimization problem remains the same so that we can combine the various elements as before to obtain the new fundamental difference equation:

$$k_{t+1} = \frac{1}{1+n}\left[\phi^A(r_{t+1})\left(w_t + Z_t^A + \frac{\theta}{1+r_{t+1}}\right) - \frac{\theta}{1+r_{t+1}}\right], \tag{4.3.6}$$

where a superscript A indicates values pertaining to the case in which consumers have access to an annuity market so that:

$$\phi^A(r_{t+1}) = \left[\frac{1+r_{t+1}}{\mu+\eta^\sigma}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{-\sigma} + 1\right]^{-1} \tag{4.3.7}$$

and the bequest transfer becomes:

$$Z_t^A = \alpha'(1+r_t)k_t + \frac{(\alpha'-1)\theta}{1+n}, \tag{4.3.8}$$

with $\alpha' \equiv \frac{\eta^\sigma}{\mu+\eta^\sigma}$. This structure highlights the similarity between the fundamental difference equation of the model without annuities. Indeed, we can write the difference equation generically as:

$$k_{t+1} = f(k_t, \mathbf{I}) = \frac{1}{1+n}\left[\phi(r_{t+1}, \mathbf{I})\left(w_t + Z_t(\mathbf{I}) + \frac{\theta}{1+r_{t+1}}\right) - \frac{\theta}{1+r_{t+1}}\right], \tag{4.3.9}$$

$$\mathbf{I} = \begin{cases} 0 & \text{without annuity market} \\ 1 & \text{with annuity market} \end{cases},$$

with this set-up in mind we may then redefine $Z_t = Z_t(0)$ and $Z_t^A = Z_t(1)$, similarly we may define $\phi(r_{t+1}) = \phi(r_{t+1}, 0)$ and $\phi^A(r_{t+1}) = \phi(r_{t+1}, 1)$. Moreover, as before, the model with annuities exhibits a unique, stable and non-trivial equilibrium at $k = k^A$.

With this structure in hand we can now turn to the core of our analysis by focusing on the impact of opening up an annuity market on steady-state capital accumulation and welfare. To this end, we can prove the following proposition.

Proposition 3. In a dynamically efficient economy for $\sigma \in (0,1)$ steady-state capital intensity and welfare are lower in the presence of an annuity market.
Proof. See Appendix C. □

Intuitively, the introduction of annuity markets crowds out unintended bequests, which leads to lower intergenerational transfers from the old to the young generation. As a consequence capital accumulation and, considering a dynamically efficient economy, welfare are reduced.

4.3.2 Numerical Example Revisited

To illustrate the results contained in Proposition 3, we return to our numerical example. To this end, we now use Table 4.1 to compare equilibrium outcomes with and without annuities. There we see that unintended bequests are lower in the presence of annuities. With the sequence of motions that this sets off we see in the final row that welfare is indeed lower in the presence of annuities.

To zoom in more on the role of bequest motives we also compare the impact of opening up an annuity market in the presence of stronger bequest motives. To this end, first consider column 3 which is the benchmark equilibrium with stronger bequest motives (i.e., higher η). Then in column 4 we introduce the annuity market. As before it leads to lower welfare and capital accumulation, but we see that magnitude of the decline is smaller (1.3 % drop versus a 2.2 % drop). In that sense bequest motives can be seen to dampen the negative general equilibrium implications of opening up an annuity market. To understand the mechanism behind this effect it serves to remember that the discrepancy between partial and general equilibrium consequences of annuitization is mainly driven by the loss of intergenerational transfers going from older to younger generations. However, with strong bequest motives individuals make less use of the annuity market – as can be seen by the lower share of total assets held in the form of annuities – and, therefore, the impact of opening up an annuity market is dampened. Nevertheless, the impact of opening up the annuity market remains a burden to individual welfare and capital accumulation.

4.4 Further Analysis

The preceding analysis has shown that bequest motives dampen the negative general equilibrium consequences of opening up an annuity market. In what follows we analyze two elements of this relationship further. First, in light of the many papers focusing on annuity market imperfections (for instance, Hansen and İmrohoroğlu (2008), Lockwood (2012) and Heijdra and Mierau (2012)), we analyze the impact of opening up a imperfect annuity market. Second, following Pecchenino and Pollard (1997) and Heijdra *et al.* (2017b) we consider the impact of limiting the amount of assets that individuals can annuitize.

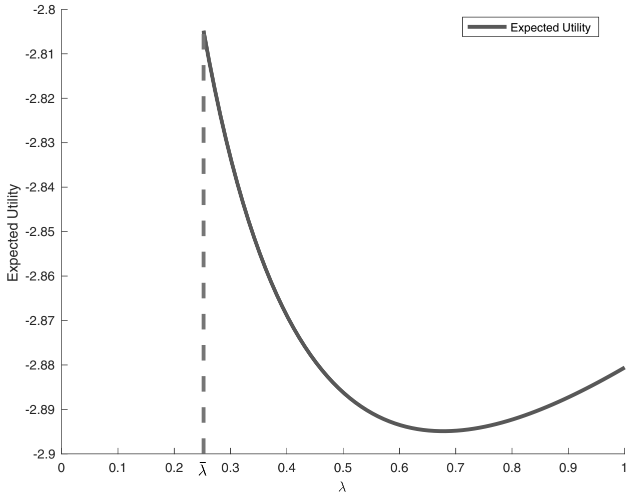
4.4.1 Imperfect Annuities

While hitherto we have considered that if an annuity market is opened, its is actuarially fair, we now consider one in which, for some reason, the annuity premium is not actuarially fair. Essentially, this implies that the annuity premium now becomes:

$$1 + r_{t+1}^{A} = \frac{1 + r_{t+1}}{\mu}(1 - (1 - \lambda)(1 - \mu)), \quad 0 < \lambda < 1, \tag{4.4.1}$$

where λ is an omnibus imperfection term that can capture numerous reasons as to why the annuity market is imperfect (see, Lockwood (2012) for a similar approach).

Figure 4.2: Imperfect Annuities



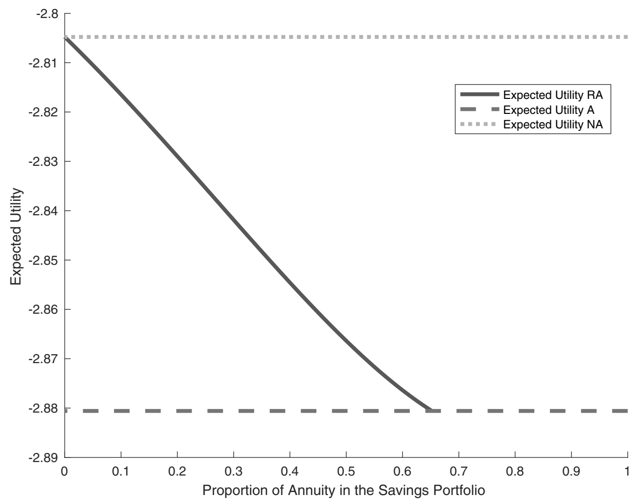
While in our companion paper (Heijdra *et al.,* 2017a) we study various micro origins of annuity market imperfection, for the current purpose we assume that, for exhibition purposes, the imperfection arises from administration costs.[4]

Leaving technicalities aside we trace out the expected utility levels from opening up an annuity market with varying degrees of imperfection in Figure 4.2. There we see that for a grossly imperfect annuity market, individuals do not annuitize at all so that there is no difference between opening up an annuity market or not. This result echoes the early analysis of Lockwood (2012) who emphasizes that the substantial difference between the actuarially fair price of annuities and the actual price of annuities goes some way in accounting for the annuity puzzle. Indeed, in the presence of bequest motives, individuals attach a positive value to their bequests so that in order to surrender them to the annuity firm they need to be compensated at a substantial margin.

Considering ever more perfect annuities we observe that utility quickly drops as individuals start annuitizing their assets, leading to the loss of accidental bequests outlined above. Eventually, however, the share of assets annuitized becomes so substantial that an increase in the annuity premium (i.e., a decrease of the imperfection) leads to a welfare increase. In sum, we observe that having access to a perfect annuity market is not necessarily better than having access to an imperfect annuity market. Because the imperfect annuity market acts so as to restrain individuals from annuitizing their assets, having a less-than-perfect annuity market diminishes the adverse general equilibrium consequences of opening up an the annuity market.

[4] In order not to allow a *leak* into the model, we allow the administration costs to be redistributed as lump-sum transfer to the young together with the bequest.

Figure 4.3: Imperfect Annuities



4.4.2 Restricted Access to Annuities

We impose a limit on the share of assets that can be annuitized and consider whether setting this limit to below what individuals would chose can lead to a welfare gain from opening up an annuity market. Following Pecchenino and Pollard (1997) and Heijdra *et al.* (2017), we restrict the share of assets that may be annuitized to $\psi \in (0, 1)$. This implies that individuals face an additional constraint:

$$A_t = \psi(A_t + S_t). \tag{4.4.2}$$

As above we trace out the welfare consequences of imposing such a constraint in Figure **??**. There we see that for values of ψ close to 1 the constraint is not binding and individuals simply chose the share of annuities as before. In line with the numerical example above we find that for sufficiently larger ψ, individuals annuitize about 65% of their assets. If, however, ψ is smaller than the share of assets that individuals voluntarily annuitze, we observe that the welfare loss from opening up an annuity market gradually, and monotonically, becomes smaller.

4.5 Conclusion

Using a tractable two-period overlapping generations model we consider the role of annuities and bequests in general equilibrium. We highlight that bequests increase capital accumulation because they effectuate an intergenerational transfer from the old to the young. Importantly, we show that also when bequest motives are taken into account, opening up an annuity market will lead to a decrease in capital accumulation and welfare. We do, however, observe that the impact of opening up the annuity market is dampened by the presence of bequest motives. Nevertheless, our results substantiate the insight that basing public policy concerning annuities on partial equilibrium models of annuitization is potentially misguided.

4.A Proof of Proposition 1

Proposition 1. There exists a unique, stable and non-trivial steady state at $k(t) = k$.

If a steady-state equilibrium of the model satisfying $k(t) = k$ exists, it is a solution to the fundamental difference equation:

$$k = \frac{1}{1+n}\left[\phi(k)\left(w(k) + Z(k) + \frac{\theta}{1+r(k)}\right) - \frac{\theta}{1+r(k)}\right]. \tag{4.A.1}$$

Rearranging (4.A.1), we have:

$$\frac{k + \frac{\theta}{(1+r(k))(1+n)}(1-\phi(k))}{\phi(k)} = \frac{w(k)+Z(k)}{1+n}, \tag{4.A.2}$$

Hence the steady-state capital intensity $k(t) = k$ is also a solution to the following equation:

$$\Omega(k) = \Gamma(k), \tag{4.A.3}$$

where $\Omega(k)$ and $\Gamma(k)$ are defined as:

$$\Omega(k) = \frac{k + \frac{\theta}{(1+r(k))(1+n)}(1-\phi(k))}{\phi(k)}, \tag{4.A.4}$$

$$\Gamma(k) = \frac{w(k)+Z(k)}{1+n}. \tag{4.A.5}$$

Let $\alpha \equiv \left[(1-\mu) + \mu\frac{\eta^\sigma}{1+\eta^\sigma}\right]$, $0 < \alpha < 1$. From Equation (4.2.15) and (4.2.16) we have:

$$\begin{aligned}
Z(k) &= \frac{(1-\mu)B^U(k) + \mu B^I(k)}{1+n},\\
Z(k) &= \alpha(1+r(k))\frac{\phi(r(k))\left[w(k) + Z(k) + \frac{\theta}{1+r(k)}\right]}{1+n} - \frac{\theta}{1+n},\\
Z(k) &= \alpha(1+r(k))k + \frac{(\alpha-1)\theta}{1+n}.
\end{aligned}$$

Assume that the threshold wealth is a fraction of the wage rate:

$$\theta = \tilde{\theta}w(k), \quad 0 < \tilde{\theta} < \min\left(1, \frac{\varepsilon\sigma(1+n)}{(1-\varepsilon)(1-\sigma)}\right). \tag{4.A.6}$$

Thus $\Gamma(k)$ is given by:

$$\Gamma(k) = \frac{\left[(1 - \frac{(1-\alpha)\tilde{\theta}}{1+n})(1-\varepsilon)A + \alpha\varepsilon A\right]k^\varepsilon + \alpha(1-\delta)k}{1+n}. \tag{4.A.7}$$

We write $\phi(k)$ in terms of capital intensity as:

$$\phi(k) = \frac{1}{(\phi_1 + k^{\varepsilon-1})^{1-\sigma}\phi_0 + 1}, \tag{4.A.8}$$

$$\phi_0 \equiv (\varepsilon A)^{1-\sigma}\alpha^{-\sigma}\frac{(1+\rho)^{\sigma}}{1+\eta^{\sigma}}, \tag{4.A.9}$$

$$\phi_1 \equiv \frac{1-\delta}{\varepsilon A}. \tag{4.A.10}$$

Lemma 4.A.1. *[Properties of the $\Gamma(k)$ function] The function $\Gamma(k)$ has the following properties:*

$$(i)\ \Gamma(0) = 0;$$

$$(ii)\ \Gamma'(k) = \frac{\varepsilon\left[(1-\frac{(1-\alpha)\tilde{\theta}}{1+n})(1-\varepsilon)A + \alpha\varepsilon A\right]k^{\varepsilon-1}}{1+n} + \frac{\alpha(1-\delta)}{1+n} > 0 \quad \text{for } k > 0;$$

$$(iii)\ \Gamma''(k) = -\frac{(1-\varepsilon)\varepsilon\left[(1-\frac{(1-\alpha)\tilde{\theta}}{1+n})(1-\varepsilon)A + \alpha\varepsilon A\right]k^{\varepsilon-2}}{1+n} < 0 \quad \text{for } k > 0;$$

$$(iv)\ \lim_{k\to 0}\Gamma'(k) = +\infty, \qquad \lim_{k\to\infty}\Gamma'(k) = \frac{\alpha(1-\delta)}{1+n}.$$

Proof.
Obvious by differentiation. ■

Lemma 4.A.2. *[Properties of the $\Omega(k)$ function] The function $\Omega(k)$ has the following properties:*

$$(i)\ \lim_{k\to 0}\Omega(k) = 0;$$

$$(ii)\ \Omega'(k) > 0;$$

$$(iii)\ \lim_{k\to 0}\frac{\Gamma'(k)}{\Omega'(k)} = +\infty;$$

$$(iv)\ \lim_{k\to\infty}\Omega'(k) = \phi_1^{1-\sigma}\phi_0 + 1.$$

Proof.
Part (i) follows directly from the definition:

$$\lim_{k\to 0}\Omega(k) = \lim_{k\to 0}\frac{k + \frac{\theta}{(1+r(k))(1+n)}(1-\phi(k))}{\phi(k)} = 0, \tag{4.A.11}$$

where we have used:

$$\lim_{k\to 0}\frac{k}{\phi(k)} = \lim_{k\to 0}k[(\phi_1 + k^{\varepsilon-1})^{1-\sigma}\phi_0 + 1] = 0, \tag{4.A.12}$$

$$\lim_{k\to 0}(1+r(k))\phi(k) = \lim_{k\to 0}\frac{\varepsilon A k^{\varepsilon-1} + 1 - \delta}{(\phi_1 + k^{\varepsilon-1})^{1-\sigma}\phi_0 + 1} = +\infty, \tag{4.A.13}$$

$$\lim_{k\to 0}\frac{\theta(1-\phi(k))}{(1+n)(1+r(k))\phi(k)} = \lim_{k\to 0}\frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon}\left(1-\frac{1}{(\phi_1+k^{\varepsilon-1})^{1-\sigma}\phi_0+1}\right)}{(1+n)\frac{\varepsilon A k^{\varepsilon-1}+1-\delta}{(\phi_1+k^{\varepsilon-1})^{1-\sigma}\phi_0+1}} = 0. \tag{4.A.14}$$

To prove part (ii) we define $\gamma(k) \equiv \frac{\theta}{1+r(k)}$, which has the following properties:

$$\gamma'(k) = \frac{\tilde{\theta}\varepsilon(1-\varepsilon)Ak^{\varepsilon-1}}{\varepsilon Ak^{\varepsilon-1}+1-\delta} + \frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon}\varepsilon(1-\varepsilon)Ak^{\varepsilon-2}}{\left(\varepsilon Ak^{\varepsilon-1}+1-\delta\right)^2} > 0, \tag{4.A.15}$$

$$\lim_{k\to 0}\frac{\gamma(k)}{k} = \lim_{k\to 0}\frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon-1}}{\varepsilon Ak^{\varepsilon-1}+1-\delta} = \frac{\tilde{\theta}(1-\varepsilon)}{\varepsilon}, \tag{4.A.16}$$

$$\lim_{k\to 0}\gamma'(k) = \lim_{k\to 0}\left(\frac{\tilde{\theta}\varepsilon(1-\varepsilon)Ak^{\varepsilon-1}}{\varepsilon Ak^{\varepsilon-1}+1-\delta} + \frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon}\varepsilon(1-\varepsilon)Ak^{\varepsilon-2}}{\left(\varepsilon Ak^{\varepsilon-1}+1-\delta\right)^2}\right) = \frac{\tilde{\theta}(1-\varepsilon)}{\varepsilon}, \tag{4.A.17}$$

$$\lim_{k\to \infty}\frac{\gamma(k)}{k} = \lim_{k\to \infty}\frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon-1}}{\varepsilon Ak^{\varepsilon-1}+1-\delta} = 0, \tag{4.A.18}$$

$$\lim_{k\to \infty}\gamma'(k) = \lim_{k\to \infty}\left(\frac{\tilde{\theta}\varepsilon(1-\varepsilon)Ak^{\varepsilon-1}}{\varepsilon Ak^{\varepsilon-1}+1-\delta} + \frac{\tilde{\theta}(1-\varepsilon)Ak^{\varepsilon}\varepsilon(1-\varepsilon)Ak^{\varepsilon-2}}{\left(\varepsilon Ak^{\varepsilon-1}+1-\delta\right)^2}\right) = 0. \tag{4.A.19}$$

We can write $\Omega(k)$ as:

$$\Omega(k) = \frac{k + \frac{\gamma(k)}{1+n}(1-\phi(k))}{\phi(k)}, \tag{4.A.20}$$

$$\begin{aligned}
\Omega'(k) &= \frac{\left(1+\frac{\gamma'(k)(1-\phi(k))}{1+n}\right)\phi(k) - \left(k+\frac{\gamma(k)}{1+n}\right)\phi'(k)}{\phi(k)^2}, \\
&= \frac{\left(1+\frac{\gamma'(k)(1-\phi(k))}{1+n}\right) - \left(1+\frac{\gamma(k)}{(1+n)k}\right)(1-\sigma)(1-\varepsilon)(1-\phi(k))\frac{1}{\phi_1 k^{1-\varepsilon}+1}}{\phi(k)}.
\end{aligned} \tag{4.A.21}$$

$$\begin{aligned}
\Omega'(k) &> \frac{\left(1+\frac{\gamma'(k)(1-\phi(k))}{1+n}\right) - \left(1+\frac{\tilde{\theta}(1-\varepsilon)}{\varepsilon(1+n)}\right)(1-\sigma)(1-\varepsilon)(1-\phi(k))\frac{1}{\phi_1 k^{1-\varepsilon}+1}}{\phi(k)}, \\
&> \frac{1-(1+\frac{\sigma}{1-\sigma})(1-\sigma)(1-\varepsilon)(1-\phi(k))\frac{1}{\phi_1 k^{1-\varepsilon}+1}}{\phi(k)}, \\
&> 0,
\end{aligned}$$

where we have used (4.A.6).

We use (4.A.21) and Lemma 4.A.1 (ii) to write:

$$\frac{\Gamma'(k)}{\Omega'(k)} = \frac{1}{1+n}\frac{\left(\varepsilon\left[\left(1-\frac{(1-\alpha)\tilde{\theta}}{1+n}\right)(1-\varepsilon)A+\alpha\varepsilon A\right]k^{\varepsilon-1}+\alpha(1-\delta)\right)\phi(k)}{\left(1+\frac{\gamma'(k)(1-\phi(k))}{1+n}\right) - \left(1+\frac{\gamma(k)}{(1+n)k}\right)(1-\sigma)(1-\varepsilon)(1-\phi(k))\frac{1}{\phi_1 k^{1-\varepsilon}+1}}. \tag{4.A.22}$$

Part (iii) follows immediately from (4.A.16), (4.A.17) and:

$$\lim_{k\to 0} \phi(k) = \lim_{k\to 0} \frac{1}{\left(\phi_1 + k^{\varepsilon-1}\right)^{1-\sigma} \phi_0 + 1} = 0, \tag{4.A.23}$$

$$\lim_{k\to 0} k^{\varepsilon-1}\phi(k) = \lim_{k\to 0} \frac{1}{\left(\phi_1 k^{\frac{1-\varepsilon}{1-\sigma}} + k^{\frac{\sigma(1-\varepsilon)}{1-\sigma}}\right)^{1-\sigma} \phi_0 + k^{1-\varepsilon}} = +\infty. \tag{4.A.24}$$

To prove part (iv) we use (4.A.18), (4.A.19) and:

$$\lim_{k\to\infty} k^{1-\varepsilon} = +\infty, \tag{4.A.25}$$

$$\lim_{k\to\infty} \phi(k) = \lim_{k\to\infty} \frac{1}{\left(\phi_1 + k^{\varepsilon-1}\right)^{1-\sigma} \phi_0 + 1} = \frac{1}{\phi_1^{1-\sigma}\phi_0 + 1}, \tag{4.A.26}$$

thus we have:

$$\begin{aligned}\lim_{k\to\infty} \Omega'(k) &= \frac{\left(1 + \frac{\gamma'(k)(1-\phi(k))}{1+n}\right) - \left(1 + \frac{\gamma(k)}{(1+n)k}\right)(1-\sigma)(1-\varepsilon)(1-\phi(k))\frac{1}{\phi_1 k^{1-\varepsilon}+1}}{\phi(k)},\\ &= \phi_1^{1-\sigma}\phi_0 + 1. \end{aligned}\tag{4.A.27}$$

■

The steady-state equation (4.A.3) has two roots. By Lemmas 4.A.1(i) and 4.A.2(i) one root is at $k = 0$. The trivial steady-state solution is unstable:

$$\frac{dk_{t+1}}{dk_t}|_{k=0} = \lim_{k\to 0} \frac{\Gamma'(k)}{\Omega'(k)} = +\infty, \tag{4.A.28}$$

where we have used Lemma 4.A.2(iii). It follows that $\Gamma(k)$ lies above $\Omega(k)$ for positive values of k close to the origin. We know that $\Gamma(k)$ is concave while $\Omega(k)$ is strictly increasing. When k goes to infinity, they satisfy $\lim_{k\to\infty}\Gamma'(k) = \frac{\alpha(1-\delta)}{1+n} < 1 < \phi_1^{1-\sigma}\phi_0 + 1 = \lim_{k\to\infty}\Omega'(k)$. Hence there is a unique, finite nontrivial root, k. At $k(t) = k$, $\Omega(k)$ cuts $\Gamma(k)$ from below such that $0 < \Gamma'(k)/\Omega'(k) < 1$, thus proving the stability of the nontrivial equilibrium.

4.B Proof of Proposition 2

Proposition 2. Capital intensity increases with the strength of the bequest motives and decreases with threshold wealth:

$$\frac{\partial k}{\partial \eta} > 0 \quad \text{and} \quad \frac{\partial k}{\partial \theta} < 0.$$

Proof.
In this Appendix we show that $\frac{\partial k}{\partial \theta} < 0$ and $\frac{\partial k}{\partial \eta} > 0$ given that r is bounded by a certain upper limit. We differentiate the fundamental difference equation on both sides with respect to θ:

$$\frac{\partial k}{\partial \theta} = \frac{1}{1+n}\left[\phi\left(\frac{\alpha-1}{1+n}+\frac{1}{1+r}\right)-\frac{1}{1+r}\right], \tag{4.B.1}$$

$$= \frac{1}{1+n}\phi\frac{\alpha-1}{1+n}+\frac{1}{1+n}(\phi-1)\frac{1}{1+r}. \tag{4.B.2}$$

By definition, $0 < \alpha < 1$ and $0 < \phi < 1$, hence:

$$\frac{\partial k}{\partial \theta} < 0. \tag{4.B.3}$$

From the fundamental difference equation we differentiate both sides with respect to η:

$$\frac{\partial k}{\partial \eta} = \frac{1}{1+n}\left[\frac{\partial \phi}{\partial \eta}\left(w+Z+\frac{\theta}{1+r}\right)+\phi\left(\frac{\partial w}{\partial \eta}+\frac{\partial Z}{\partial \eta}-\frac{\theta}{(1+r)^2}\frac{\partial r}{\partial \eta}\right)+\frac{\theta}{(1+r)^2}\frac{\partial r}{\partial \eta}\right]. \tag{4.B.4}$$

From the definition of r, w and Z, we have:

$$\frac{\partial r}{\partial \eta} = \varepsilon(\varepsilon-1)\Omega_o k^{\varepsilon-2}\frac{\partial k}{\partial \eta} = (\varepsilon-1)\frac{r+\delta}{k}\frac{\partial k}{\partial \eta}, \tag{4.B.5}$$

$$\frac{\partial w}{\partial \eta} = \varepsilon(1-\varepsilon)\Omega_o k^{\varepsilon-1}\frac{\partial k}{\partial \eta} = (1-\varepsilon)(r+\delta)\frac{\partial k}{\partial \eta}, \tag{4.B.6}$$

$$\frac{\partial Z}{\partial \eta} = \alpha(1+r)\frac{\partial k}{\partial \eta}+\alpha k\frac{\partial r}{\partial k} = \alpha(1+r)\frac{\partial k}{\partial \eta}+\alpha(\varepsilon-1)(r+\delta)\frac{\partial k}{\partial \eta}. \tag{4.B.7}$$

Substituting Eq. (4.B.5) – (4.B.7) into Eq.(4.B.4) and rearranging, we obtain:

$$\frac{\partial k}{\partial \eta} = \frac{w+Z+\frac{\theta}{1+r}}{(1+n)-\phi\left[(1-\alpha)(1-\varepsilon)(r+\delta)+\alpha(1+r)\right]+(1-\phi)(1-\varepsilon)\frac{\theta(r+\delta)}{(1+r)^2 k}}\frac{\partial \phi}{\partial \eta}, \tag{4.B.8}$$

We observe that $\frac{\partial \phi}{\partial \eta} > 0$. As long as the denominator is positive, $\frac{\partial k}{\partial \eta}$ is larger than 0. A sufficient condition for $\frac{\partial k}{\partial \eta} > 0$ to hold is:

$$r < \frac{\frac{1+n}{\phi}-(1-\alpha)(1-\varepsilon)\delta-\alpha}{(1-\alpha)(1-\varepsilon)+\alpha}. \tag{4.B.9}$$

■

4.C Proof of Proposition 3

Proposition 3. In a dynamically efficient economy for $\sigma \in (0,1)$ steady-state capital intensity and welfare are lower in the presence of an annuity market.

First we show that in a dynamically efficient economy steady-state capital intensity is lower in the presence of an annuity market.

Proof.
From Eq. (4.3.9) the steady-state capital intensity in each case is determined by:

$$k = f(k, \mathbf{I}) = \frac{1}{1+n}\frac{1}{1+r}\left[\phi(r, \mathbf{I})\left(w + Z(\mathbf{I}) + \frac{\theta}{1+r}\right) - \frac{\theta}{1+r}\right], \tag{4.C.1}$$

$$\mathbf{I} = \begin{cases} 0 & \text{without annuity market} \\ 1 & \text{with annuity market} \end{cases}$$

where $\phi(r, \mathbf{I})$ is the marginal propensity to save:

$$\phi(r, 0) = \left[\frac{1+r}{1+\eta^\sigma}\left(\frac{1+r}{1+\rho}\left[\left(\frac{\eta^\sigma}{1+\eta^\sigma}\right)^{1/\sigma}(1-\mu) + \mu\right]\right)^{-\sigma} + 1\right]^{-1}, \tag{4.C.2}$$

$$\phi(r, 1) = \left[\frac{1+r}{\mu+\eta^\sigma}\left(\frac{1+r}{1+\rho}\right)^{-\sigma} + 1\right]^{-1}. \tag{4.C.3}$$

$Z(\mathbf{I})$ is the intergenerational government transfer:

$$Z(0) = \alpha(1 + r(k_t))k_t + \frac{(\alpha - 1)\theta}{1+n}, \quad \alpha \equiv \left[(1-\mu) + \mu\frac{\eta^\sigma}{1+\eta^\sigma}\right], \tag{4.C.4}$$

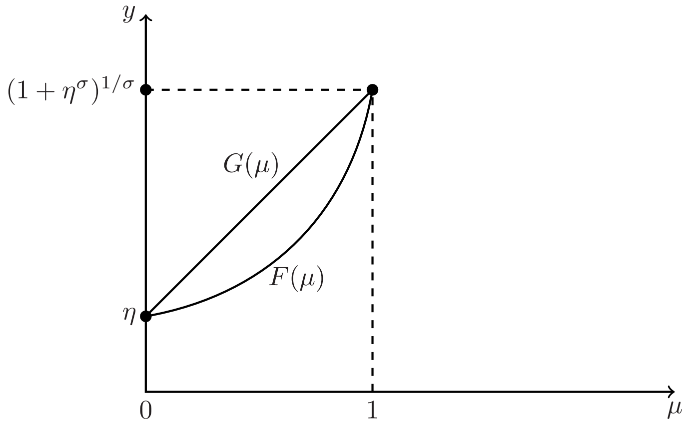
$$Z(1) = \alpha'(1 + r(k_t))k_t + \frac{(\alpha' - 1)\theta}{1+n}, \quad \alpha' \equiv \frac{\eta^\sigma}{\mu+\eta^\sigma}. \tag{4.C.5}$$

Taking total derivative of 4.C.1 at $k(t) = k$ and rearranging, we obtain:

$$\left.\frac{dk}{d\mathbf{I}}\right|_{k(t)=k} = \frac{\frac{\partial f(k,\mathbf{I})}{\partial \mathbf{I}}}{1 - \left.\frac{\partial f(k,\mathbf{I})}{\partial k}\right|_{k(t)=k}}. \tag{4.C.6}$$

From Appendix A, we know that the nontrivial equilibrium is stable, $\left.\frac{\partial f(k,\mathbf{I})}{\partial k}\right|_{k(t)=k} < 1$. To prove $\left.\frac{dk}{d\mathbf{I}}\right|_{k(t)=k} < 0$, we only need to show that $\frac{\partial f(k,\mathbf{I})}{\partial \mathbf{I}} < 0$. In the following we show that $\frac{\partial f(k,\mathbf{I})}{\partial \mathbf{I}} < 0$ when $0 < \sigma < 1$.

Figure 4.4: Function $F(\mu)$ and $G(\mu)$



First, we show that for any given k, $\phi(r,0) > \phi(r,1)$:

$$\begin{aligned}\phi(r,0) > \phi(r,1) &\Leftarrow \frac{1}{1+\eta^\sigma}\left[\left(\frac{\eta^\sigma}{1+\eta^\sigma}\right)^{1/\sigma}(1-\mu)+\mu\right]^{-\sigma} < \frac{1}{\mu+\eta^\sigma} \\ &\Leftarrow \left(\frac{\eta^\sigma}{1+\eta^\sigma}\right)^{1/\sigma}(1-\mu)+\mu > \left(\frac{\mu+\eta^\sigma}{1+\eta^\sigma}\right)^{1/\sigma} \\ &\Leftarrow \eta(1-\mu)+\mu(1+\eta^\sigma)^{1/\sigma} > (\mu+\eta^\sigma)^{1/\sigma} \qquad (4.C.7)\end{aligned}$$

To show that Equation (4.C.7) holds, we define $F(\mu) \equiv (\mu+\eta^\sigma)^{1/\sigma}$ and $G(\mu) \equiv \eta(1-\mu)+\mu(1+\eta^\sigma)^{1/\sigma}$ and show that $G(\mu) - F(\mu) > 0$. On the interval $\mu \in [0,1]$, we have:

$$F(0) = G(0) = \eta, \qquad (4.C.8)$$

$$F(1) = G(1) = (1+\eta^\sigma)^{1/\sigma}, \qquad (4.C.9)$$

$$F'(\mu) = \frac{1}{\sigma}(\mu+\eta^\sigma)^{1/\sigma-1} > 0, \quad F''(\mu) = \left(\frac{1}{\sigma}-1\right)\frac{1}{\sigma}(\mu+\eta^\sigma)^{1/\sigma-2} > 0, \qquad (4.C.10)$$

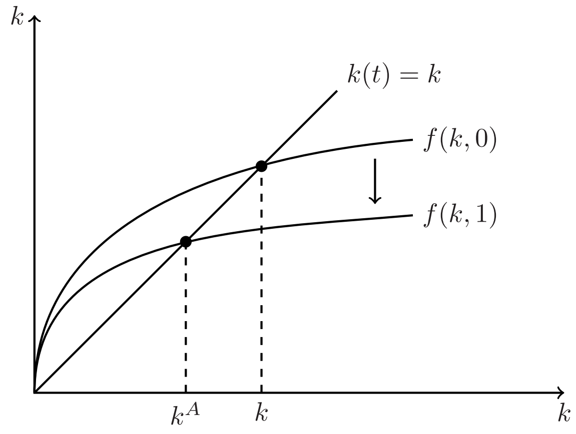
$$G'(\mu) = (1+\eta^\sigma)^{1/\sigma} - \eta > 0, \quad G''(\mu) = 0. \qquad (4.C.11)$$

$G(\mu)$ is strictly increasing and linear in μ while $F(\mu)$ is strictly increasing and convex in μ on the interval $\mu \in [0,1]$. From Figure 4.4 we see that $G(\mu) > F(\mu)$, when $\mu \in (0,1)$.

Then we show that for any given k, $Z(0) > Z(1)$:

$$Z(0) > Z(1) \Leftarrow \alpha > \alpha' \qquad (4.C.12)$$

Figure 4.5: Steady-state Capital Intensity



From Equations (4.C.4) and (4.C.5) we have:

$$\begin{aligned}
\alpha - \alpha' &= (1-\mu) + \mu\frac{\eta^\sigma}{1+\eta^\sigma} - \frac{\eta^\sigma}{\mu+\eta^\sigma}, \\
&= \frac{[1-\mu+\eta^\sigma]\,(\mu+\eta^\sigma) - (1+\eta^\sigma)\,\eta^\sigma}{(1+\eta^\sigma)\,(\mu+\eta^\sigma)}, \\
&= \frac{\mu(1-\mu)}{(1+\eta^\sigma)(\mu+\eta^\sigma)}. \qquad (4.C.13)
\end{aligned}$$

$\alpha > \alpha'$ holds when $\mu \in (0,1)$.

Since $\phi(r,0) > \phi(r,1)$ and $Z(0) > Z(1)$ for any given k, we have $\frac{\partial f(k,\mathbf{I})}{\partial \mathbf{I}} < 0$ when $0 < \sigma < 1$, $0 < \mu < 1$. As illustrated in Figure 4.5, introducing the annuity market rotates function f clockwise, which pins down a lower capital intensity in the steady state k^A. ■

Now we are ready to show that in a dynamically efficient economy welfare is lower in the presence of an annuity market.

Proof.
We employ the Envelope Theorem to analyze the long-run welfare effects. The Lagrangian for the constrained welfare optimization is given by:

$$\begin{aligned}
\mathcal{L}(\mathbf{I}) \equiv{}& U(C^Y) + \frac{1-\mu}{1+\rho}V(B^U) + \frac{\mu}{1+\rho}\left(U(C^O) + V(B^I)\right) \\
&- \lambda\left[C^Y + \frac{C^O[1-\mathbf{I}(1-\mu)]}{1+r} + \frac{B^I[1-\mathbf{I}(1-\mu)]}{1+r} + \frac{\mathbf{I}(1-\mu)}{1+r}B^A - W - Z\right] \qquad (4.C.14)
\end{aligned}$$

The first order conditions are:

$$U'(C^Y) = \lambda \tag{4.C.15}$$

$$\frac{\mu}{1+\rho}U'(C^O) = \lambda\frac{1-\mathbf{I}(1-\mu)}{1+r} \tag{4.C.16}$$

$$\frac{\mu}{1+\rho}V'(B^I) = \lambda\frac{1-\mathbf{I}(1-\mu)}{1+r} \tag{4.C.17}$$

$$W + Z - \frac{\mathbf{I}(1-\mu)}{1+r}B^U = C^Y + \frac{C^O[1-\mathbf{I}(1-\mu)]}{1+r} + \frac{B^I[1-\mathbf{I}(1-\mu)]}{1+r} \tag{4.C.18}$$

By employing the Envelope Theorem, we have:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda}{d\mathbf{I}} &= \frac{\mathcal{L}(\mathbf{I})}{\partial\mathbf{I}} \\
&= U'(C^Y)\frac{\partial C^Y}{\partial\mathbf{I}} + \frac{1-\mu}{1+\rho}V'(B^U)\frac{\partial B^U}{\partial\mathbf{I}} + \frac{\mu}{1+\rho}U'(C^O)\frac{\partial C^O}{\partial\mathbf{I}} + \frac{\mu}{1+\rho}V'(B^I)\frac{\partial B^I}{\partial\mathbf{I}} \\
&\quad + \left[C^Y + \frac{C^O[1-\mathbf{I}(1-\mu)]}{1+r} + \frac{B^I[1-\mathbf{I}(1-\mu)]}{1+r} + \frac{\mathbf{I}(1-\mu)}{1+r}B^U - W - Z\right]\frac{\partial\lambda}{\partial\mathbf{I}} \\
&\quad - \lambda\left[\frac{\partial C^Y}{\partial\mathbf{I}} + \frac{1-\mathbf{I}(1-\mu)}{1+r}\frac{\partial C^O}{\partial\mathbf{I}} - \frac{(1-\mu)C^O}{1+r} - \frac{C^O[1-\mathbf{I}(1-\mu)]}{(1+r)^2}\frac{\partial r}{\partial\mathbf{I}}\right. \\
&\quad \left. + \frac{1-\mathbf{I}(1-\mu)}{1+r}\frac{\partial B^I}{\partial\mathbf{I}} - \frac{(1-\mu)B^I}{1+r} - \frac{B^I[1-\mathbf{I}(1-\mu)]}{(1+r)^2}\frac{\partial r}{\partial\mathbf{I}} + \frac{1-\mu}{1+r}B^U - \frac{\partial w}{\partial\mathbf{I}} - \frac{\partial Z}{\partial\mathbf{I}}\right].
\end{aligned} \tag{4.C.19}$$

We start from the steady state of the case without annuity market, such that $\mathbf{I} = 0$. Substituting the first order conditions (4.C.15) – (4.C.18) into (4.C.19) and evaluating at $\mathbf{I} = 0$ gives:

$$\frac{d\mathbb{E}\Lambda}{d\mathbf{I}} = \frac{1-\mu}{1+\rho}V'(B^U)\frac{\partial B^U}{\partial\mathbf{I}} + \lambda\frac{\partial Z}{\partial\mathbf{I}} + \lambda\left[\frac{C^O+B^I}{(1+r)^2}\frac{\partial r}{\partial\mathbf{I}} + \frac{\partial w}{\partial\mathbf{I}}\right]. \tag{4.C.20}$$

In the steady state of the case without annuity market, we have

$$\frac{\partial r}{\partial\mathbf{I}} = \varepsilon(\varepsilon-1)\Omega_o k^{\varepsilon-2} = (\varepsilon-1)\frac{r+\delta}{k}\frac{\partial k}{\partial\mathbf{I}}, \tag{4.C.21}$$

$$\frac{\partial w}{\partial\mathbf{I}} = \varepsilon(1-\varepsilon)\Omega_o k^{\varepsilon-1} = (1-\varepsilon)(r+\delta)\frac{\partial k}{\partial\mathbf{I}}, \tag{4.C.22}$$

$$C^O + B^I = (1+r)S = (1+r)(1+n)k. \tag{4.C.23}$$

We can rewrite (45) by using (46) – (48) to obtain:

$$\frac{d\mathbb{E}\Lambda}{d\mathbf{I}} = \frac{1-\mu}{1+\rho}V'(B^U)\frac{\partial B^U}{\partial\mathbf{I}} + \lambda\frac{\partial Z}{\partial\mathbf{I}} + U'(C^Y)\Delta\frac{\partial k}{\partial\mathbf{I}}, \tag{4.C.24}$$

where $\Delta = \frac{(1-\varepsilon)(r-n)(r+\delta)}{1+r} > 0$.

The annuity market crowds out unintended bequest ($\frac{\partial B^U}{\partial\mathbf{I}} < 0$). As shown above, it also crowds out intergenerational wealth transfer and capital intensity ($\frac{\partial Z}{\partial\mathbf{I}} < 0$,

$\frac{\partial k}{\partial \mathbf{I}} < 0$). Hence the individual welfare unambiguously falls with the introduction of annuity market:

$$\frac{d\mathbb{E}\Lambda}{d\mathbf{I}} < 0. \tag{4.C.25}$$

$\blacksquare$

Chapter 5

Socially Optimal Fertility*

*This chapter is based on Jiang (2019).

5.1 Introduction

It is not hard to observe that there exists a positive externality of fertility on the old age consumption as long as there is intergenerational transfer from the young to the old. In an extreme case, as Samuelson (1958) assumes, where nothing will keep at all in the future, the old generation would die at the beginning of their retirement years if they fail to bribe the younger generation to support them. But if they succeed in doing so, a higher fertility rate across the society would mostly likely benefit the old generation. Meanwhile, Neoclassical growth theory predicts a negative externality of fertility on the youth consumption, since increased population growth requires a larger fraction of output to be invested as capital if the capital per capita is to be maintained. Observing the two opposite effects of '*intergenerational transfer*' and '*capital dilution*', Samuelson (1975) proposed his celebrated *Serendipity theorem*,[1] which shows the possibility of an interior optimal rate of population growth.

However, Samuelson's solution, as pointed out by Deardorff (1976), is *not* optimal in general. By using Cobb-Douglas utility and production functions, Deardorff (1976) showed that Samuelson's solution provides a global *minimum* of steady-state utility instead of a maximum. But he also admitted that Samuelson's optimal rate of growth of population may indeed be optimal for sufficiently inelastic production functions. Later Michel and Pestieau (1993) established conditions that imply an interior solution. For CES utility and production functions with low elasticity of substitution, the serendipity theorem holds.

Recent researches have paid more attention to the role of individual fertility decisions in determining the optimal fertility rate and intergenerational transfer scheme. Van Groezen et al. (2003) have shown that a pay-as-you-go pension system combined with a child allowance scheme can lead to the social optimum. They argue that the government needs a child allowance scheme to alter the number of offspring because it wants to redistribute to the old through the pay-as-you-go pension system, where the externalities of endogenous fertilities arise. Through the child allowance scheme, the externalities of endogenous fertilities can be fully internalized. Abío et al. (2004) claim that they have found the silver bullet—a pay-as-you-go pension system linking pension benefits to the number of children—to induce both the optimal capital stock and the optimal rate of population growth. Their inspiration comes from the observation that 'in most countries the system promises a pension which is solely dependent on the evolution in wages and not on demographic evolution.' They believe that this *single* policy alone can lead the economy to reach the first-best outcome.

This Chapter is most closely related to the work by Michel and Pestieau (1993). We extend their work by assuming that fertility choices are *endogenous* rather than exogenous. After all, it is more coincidental than general that the economy picks the

[1] Samuelson (1975) shows in his Serendipity Theorem that when the population growth rate is at its optimal value, competitive equilibrium can bring the first-best outcome. The first best outcome is called *the most-golden golden-rule state*, where both population growth rate and capital stock reach their optimum.

optimal growth rate and reaches the 'most-golden golden rule' lifetime state. In this Chapter, individuals are assumed to derive direct utility from their offspring and are free to choose the number of children they want subject to their budget constraints. Another extension is that we introduce a welfare concept that postulates a social welfare function that is dynamically consistent instead of Samuelson's steady-state welfare concept. We care not only about the steady-state welfare, but also about the transitional dynamics of the economy. The dynamically consistent social welfare function allows us to see clearly how the economy could be transformed, with appropriate design of government taxes, from the *Laissez faire* market economy to the first best social optimum. Following Van Groezen et al. (2003) and Abío et al. (2004), we combine a child tax and an intergenerational transfer (tax) to fix the two externalities of endogenous fertility: the '*capital dilution*' effect and the '*intergenerational transfer*' effect. The socially optimal capital stock and fertility rate are determined simultaneously after the implementation of the tax policies. Our analysis is one step further based on previous research. In Van Groezen et al. (2003) and Abío et al. (2004), they assume that the interest rate r is exogenously given, which precludes any 'capital dilution' effect. It is based on this assumption that they show the laissez faire market outcome without any government intervention is efficient. We relax this assumption and allow r to be endogenously determined by the market. Furthermore, they attribute the externality of fertility to government transfers across generations. This paper, however, takes government transfers as a means to restore individual's incentives to save, which could potentially fix the externality of 'capital dilution' effect.

Our main findings are as follows. Firstly, the unmanaged market economy will generally *not* end up in the 'most-golden golden rule' lifetime state. A plausibly calibrated version of the model reveals that individuals save less than the socially optimal capital stock as they ignore the '*capital dilution*' effect. Furthermore, the number of children they raise is more than socially optimal as they do not consider the externalities of fertility: the '*intergenerational transfer*' effect begs them to raise more children whilst the '*capital dilution*' effect requires them to make fewer babies.[2] Second, with appropriate design of government taxes, the managed market economy could reach the first best social optimum. Through an intergenerational transfer tax, the government induces the young to save more in case that the capital should be diluted. Combining with a child tax, which leads individuals to internalize the social externalities of fertility, the first best social optimum is decentralized by the managed market economy. Third, the transition from an unmanaged market economy to the first best social optimum is an improvement for all generations except the old generation at the shock. The capital per capita is predetermined at impact. After the implementation of government taxes it rises monotonically to the new steady state in about 4 to 5 periods (120 - 150 years). During the same time the fertility rate is induced to decrease and settle at the new steady state by the child tax. Most importantly, after the first period, the individual welfare level has ever been increas-

[2] In our calibration of the model the latter effect dominates the former.

ing during the transition. All generations except the old generation at the shock will benefit from the implementation of government taxes to transform the unmanaged market economy to the first best social optimum.

The rest of the Chapter is organized as follows. Section 2 describes the laissez faire market economy where we introduce the endogenous fertility. The benchmark model is calibrated with plausible structural parameters. Section 3 analyzes the social welfare of the economy. By solving the social planner's problem, we compare two concepts of social optimization: the Samuelson Social Welfare and Social Welfare Function. The latter is preferred since it considers all generations instead of the steady state generation only. Section 4 decentralizes the first best social optimum from managed market economy with government taxes. Not only do people enjoy a higher welfare level in the first best social optimum in the steady state, the transitional dynamics shows us that the transitional process is also an improvement in the long run. Section 5 concludes.

5.2 Model

We consider a two-period overlapping generations model, where in each period ($t = 1, 2, 3...$), the economy consists of a younger ('youth', superscript y) and an older ('old age', superscript o) generation. At time t the economy consists of N_t young individuals. We assume that all individuals are identical. Individuals, over their life cycle, must decide how many children to raise, how much to consume and how much to save for retirement. In the production sector, we assume that there is a large number of firms which produce homogeneous goods in a perfectly competitive market.

5.2.1 Consumers

All individuals are identical (symmetric case) so we describe the representative individual. At time t, an individual must decide on the number of children (n_t), how much to consume (c_t^y) and how much to save (s_t) in the beginning of the first period. In youth she is endowed with one unit of time, which she distributes between working ($1 - zn_t$) and raising children (zn_t). In old age she works part time as a result of mandatory retirement (λ). Her old-age consumption (c_{t+1}^o) comes from both her savings and her old-age wages. The periodic budget constraints are given by:

$$c_t^y + s_t = w_t(1 - zn_t), \tag{5.2.1}$$

$$c_{t+1}^o = (1 + r_{t+1})s_t + \lambda w_{t+1}, \tag{5.2.2}$$

where $1 + r_{t+1}$ is the rate of return on her savings, w_t and w_{t+1} are her wages during youth and old age, respectively.

We assume that an individual not only derives utility from her consumption, but also from her offspring.[3] The lifetime utility of an individual with n_t children is given

[3]For utility functions that include the quality and quantity of children, see Moav (2005).

by:

$$\Lambda_t^y \equiv V(n_t) + U(c_t^y, c_{t+1}^o), \tag{5.2.3}$$

where utility derived from goods consumption is given by the felicity function $U(\cdot)$:

$$U(c_t^y, c_{t+1}^o) \equiv \begin{cases} (c_t^y)^\zeta (c_{t+1}^o)^{1-\zeta}, & \text{for } \sigma = 1 \\ \left[\zeta (c_t^y)^{1-1/\sigma} + (1-\zeta)(c_{t+1}^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}, & \text{for } \sigma \neq 1 \end{cases} \tag{5.2.4}$$

and the weights are defined as:

$$\zeta \equiv \frac{1}{1+\beta}, \quad 1-\zeta \equiv \frac{\beta}{1+\beta} \tag{5.2.5}$$

where β is the individual's time preference discount factor ($0 < \beta < 1$). Utility derived from fertility is given by the felicity function $V(\cdot)$[4]:

$$V(n_t) \equiv \frac{\gamma}{\theta}\left[n_t - n_{min}\right]^\theta, \quad 0 < \theta < 1, \; n_{min} > 1 - \delta. \tag{5.2.7}$$

This functional form incorporates the assumption that a minimum amount of offspring is necessary to each individual. For a visualization of the felicity function, see Figure 5.1. Note that preferences are additively separable in n_t and (c_t^y, c_{t+1}^o).

5.2.2 Production

We assume perfection competition in the goods market. The production technology is represented by the following function[5]:

$$y_t = \begin{cases} \Omega k_t^\alpha l_t^{1-\alpha}, & \text{for } \xi = 1 \\ \Omega \left[\alpha k_t^{1-1/\xi} + (1-\alpha) l_t^{1-1/\xi}\right]^{1/(1-1/\xi)}, & \text{for } \xi \neq 1 \end{cases} \tag{5.2.8}$$

where Ω is total factor productivity, α is the capital share of output, $k_t \equiv \frac{K_t}{N_t}$ and $y_t \equiv \frac{Y_t}{N_t}$ are capital and output per capita, respectively. The intensive form of labor supply is defined as:

$$l_t \equiv 1 - z n_t + \frac{\lambda}{n_{t-1}}. \tag{5.2.9}$$

Firms choose the amount of labor supply and capital stock to maximize the profit. The aggregate profit is given by:

$$\Pi_t \equiv \Omega \left[\alpha K_t^{1-1/\xi} + (1-\alpha) L_t^{1-1/\xi}\right]^{1/(1-1/\xi)} - (r_t + \delta) K_t - w_t L_t. \tag{5.2.10}$$

---

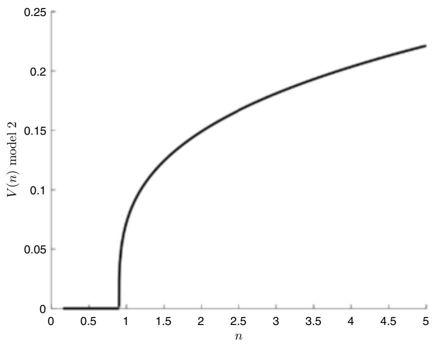
[4]We choose this function because it helps in avoiding corner solutions that result from the iso-elastic function (especially if $\lambda > 0$):

$$V(n_t) \equiv \gamma \frac{n_t^{1-1/\theta} - 1}{1 - 1/\theta}. \tag{5.2.6}$$

[5]Since the weights in preferences and technology add up to unity, the CES function converges to the Cobb-Douglas specification (both for $\sigma = 1$ and $\xi = 1$).

Figure 5.1: Utility from offspring

Minimum desired felicity



Note This is a power function featuring a minimum desired level of offspring, $n_{\min} > 0$.

Simplifying the first-order conditions, the factor prices are given by (expressed per young individual):

$$r_t + \delta = \alpha\Omega^{(\xi-1)/\xi}\left(\frac{y_t}{k_t}\right)^{1/\xi}, \tag{5.2.11}$$

$$w_t = (1-\alpha)\Omega^{(\xi-1)/\xi}\left(\frac{y_t}{l_t}\right)^{1/\xi}. \tag{5.2.12}$$

5.2.3 Market Equilibrium

This model is completed by a description of the capital market equilibrium:

$$s_t = n_t k_{t+1}, \tag{5.2.13}$$

where equation (5.2.13) says that capital next period comes from savings in the current period, which is the full income left after consumption and time cost of child:

$$s_t = w_t - (zw_t n_t + c_t^y). \tag{5.2.14}$$

In a closed economy factor prices will ensure that the capital stock is strictly positive, i.e. $s_t > 0$ and we can consolidate the budget constraints:

$$c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} + zw_t n_t = H_t^y, \tag{5.2.15}$$

where H_t^y is human wealth at birth:

$$H_t^y \equiv w_t + \lambda \frac{w_{t+1}}{1+r_{t+1}}. \tag{5.2.16}$$

Individuals choose c_t^y, c_{t+1}^o and n_t to maximize the lifetime utility (5.2.3) subject to budget constraints (5.2.15). The Lagrangian is given by:

$$\mathcal{L}^h \equiv V(n_t) + \left[\zeta (c_t^y)^{1-1/\sigma} + (1-\zeta)(c_{t+1}^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)} \tag{5.2.17}$$

$$+ \lambda_t^y \left[H_t^y - c_t^y - \frac{c_{t+1}^o}{1+r_{t+1}} - z w_t n_t\right], \tag{5.2.18}$$

we obtain the first-order conditions:

$$\frac{\partial \mathcal{L}^h}{\partial n_t} = V'(n_t) - \lambda_t^y z w_t = 0, \tag{5.2.19}$$

$$\frac{\partial \mathcal{L}^h}{\partial c_t^y} = \frac{\zeta x_t}{(c_t^y)^{1/\sigma}} - \lambda_t^y = 0, \tag{5.2.20}$$

$$\frac{\partial \mathcal{L}^h}{\partial c_{t+1}^o} = \frac{(1-\zeta) x_t}{(c_{t+1}^o)^{1/\sigma}} - \frac{\lambda_t^y}{1+r_{t+1}} = 0. \tag{5.2.21}$$

where x_t is defined as:

$$x_t \equiv \left[\zeta (c_t^y)^{1-1/\sigma} + (1-\zeta)(c_{t+1}^o)^{1-1/\sigma}\right]^{1/(\sigma-1)}. \tag{5.2.22}$$

From equations (5.2.19)–(5.2.21) and the definition of felicity function $V(\cdot)$ we obtain the optimality conditions:

$$\frac{c_{t+1}^o}{c_t^y} = \left[\beta(1+r_{t+1})\right]^\sigma, \tag{5.2.23}$$

$$\frac{\gamma [n - n_{min}]^{\theta-1} (c_t^y)^{1/\sigma}}{\zeta x_t} = z w_t. \tag{5.2.24}$$

Intuitively, equation (5.2.23) is the Euler equation. Substituting the Euler equation into x_t, we can solve for the fertility decision from Equation (5.2.24):

$$n_t = n_{min} + \left(\frac{\gamma(1+\beta)}{z w_t}\right)^{1/(1-\theta)} \left[\frac{1+\beta^\sigma (1+r_{t+1})^{\sigma-1}}{1+\beta}\right]^{1/[(\theta-1)(\sigma-1)]}. \tag{5.2.25}$$

Note that there is no wealth effect in the fertility decision. When the time cost of children or the rate of return form savings increase, parents are going to raise less children. It is easy to see that $\partial n_t / \partial (z w_t) < 0$ and $\partial n_t / \partial r_{t+1} < 0$ regardless of the magnitude of σ. The market equilibrium model is summarized in Table 5.1.

5.2.4 Parameterization Market Equilibrium

To visualize the main features of the economy, we parameterize the model by choosing plausible values for structural parameters. Here we calibrate the full version of

Table 5.1: Market equilibrium in the fertility model

$$
\begin{aligned}
s_t &= w_t - (z w_t n_t + c_t^y) \\
s_t &= n_t k_{t+1} \\
c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} + z w_t n_t &= w_t + \lambda \frac{w_{t+1}}{1 + r_{t+1}} \\
\frac{c_{t+1}^o}{c_t^y} &= \left[\beta \left(1 + r_{t+1}\right)\right]^\sigma \\
n_t &= n_{\min} + \left(\frac{\gamma(1+\beta)}{z w_t}\right)^{1/(1-\theta)} \left[\frac{1 + \beta^\sigma (1 + r_{t+1})^{\sigma - 1}}{1 + \beta}\right]^{1/[(\theta-1)(\sigma-1)]} \\
w_t &= (1 - \alpha)\, \Omega^{(\xi - 1)/\xi} \left(\frac{y_t}{l_t}\right)^{1/\xi} \\
r_t + \delta &= \alpha \Omega^{(\xi-1)/\xi} \left(\frac{y_t}{k_t}\right)^{1/\xi} \\
y_t &= \Omega \left[\alpha k_t^{1-1/\xi} + (1-\alpha) l_t^{1-1/\xi}\right]^{1/(1-1/\xi)} \\
l_t &= 1 - z n_t + \frac{\lambda}{n_{t-1}}
\end{aligned}
$$

Note Variables: saving of the young s_t, fertility rate n_t, labour supply l_t, wage rate w_t, real interest rate r_t, current capital stock k_t, youth consumption c_t^y, (planned) old-age consumption c_{t+1}^o, output y_t. Individuals are endowed with perfect foresight regarding future variables.

the model as stated in Table 5.1. Hence, we assume that $0 < \delta < 1$, $z > 0$, $\lambda > 0$ and $\gamma > 0$. First, we postulate plausible values for the substitution elasticity in consumption and production ($\sigma = 0.7$, $\xi = 0.7$), the annual capital depreciation rate ($\delta_a = 0.06$), the annual growth rate of the population ($n_a = 0.01$), the social annual rate of time preference ($\rho_{G,a} = 0.02$) and the target annual interest rate ($\hat{r}_a = 0.05$). The output is normalized at unity: $\hat{y} = 1$. The output share of labour income is targeted at $\hat{w}\hat{l}/\hat{y} = \hat{\omega}_L = 0.7$. The minimum desired offspring is set at $n_{min} = 0.9$. Second, we set the length of each period to be 30 years and compute the values for $\hat{n}$, $\hat{r}$ and δ.[6] The time cost of fertility is set at $z\hat{n} = \phi_n = 0.2$ (six years). Labour supply per individual is $\hat{l} = 1 - \phi_n + \frac{\lambda}{\hat{n}}$. Finally, we can compute $(\hat{k}, \alpha, \Omega)$ from the competitive conditions for capital demand, labour share and output for a given value of ξ:

$$\hat{r} + \delta = f_1(\hat{k}, \hat{l}) = \alpha\Omega^{(\xi-1)/\xi} \left(\frac{\hat{y}}{\hat{k}}\right)^{1/\xi}, \tag{5.2.26}$$

$$\hat{\omega}_L = (1-\alpha)\Omega^{(\xi-1)/\xi} \left(\frac{\hat{y}}{\hat{l}}\right)^{(1-\xi)/\xi}, \tag{5.2.27}$$

$$\hat{y} = \Omega \left[\alpha\hat{k}^{1-1/\xi} + (1-\alpha)\hat{l}^{1-1/\xi}\right]^{1/(1-1/\xi)}. \tag{5.2.28}$$

The resulting structural parameters are reported in Table 5.2. With the structural parameters we are able to compute the wage rate, savings and consumption:

$$\hat{w} = f(\hat{k}, \hat{l}) - \hat{k}f_2(\hat{k}, \hat{l}) = (1-\alpha)\Omega^{(\xi-1)/\xi} \left(\frac{\hat{y}}{\hat{l}}\right)^{1/\xi}, \tag{5.2.29}$$

$$\hat{s} = \hat{n}\hat{k}, \tag{5.2.30}$$

$$\hat{c}^y = \hat{w}(1 - z\hat{n}) - \hat{s}, \tag{5.2.31}$$

$$\hat{c}^o = (1+\hat{r})\hat{s} + \lambda\hat{w}. \tag{5.2.32}$$

From the Euler equation we can solve for β:

$$\beta = \frac{1}{1+\hat{r}} \left(\frac{\hat{c}^o}{\hat{c}^y}\right)^{1/\sigma}. \tag{5.2.33}$$

The child preference parameter γ is used as a calibration parameter, i.e. it is set at the value such that in the steady state the fertility demand is satisfied:

$$\gamma = \frac{z\hat{w}}{1+\beta}(\hat{n} - n_{min})^{1-\theta} \left[\frac{1+\beta^\sigma(1+\hat{r})^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. \tag{5.2.34}$$

[6] Note that $\hat{n} = (1+n_a)^{30} = 1.3478$, $\hat{r} = (1+r_a)^{30} - 1 = 3.3219$ and $\delta = 1-(1-\delta_a)^{30} = 0.8437$.

Table 5.2: Structural parameters: General case

σ	substitution elasticity in consumption		0.7000
ξ	substitution elasticity in production		0.7000
α	capital efficiency parameter		0.1148
β	private time preference parameter	c	0.5722
ρ_a	pure annual rate of time preference (percent)	i	1.8783
β_G	social discount factor		0.5521
$n_{\min}$	minimum desired offspring		0.9000
δ_a	annual capital depreciation rate (percent)		6.0000
δ	capital depreciation factor	i	0.8437
Ω	scale factor production function	c	1.4767
γ	taste for offspring parameter	c	0.0435
θ	curvature coefficient for children		0.3000
z	time cost of children		0.1484
λ	fraction of time worked in old age		0.5000

Note The parameters labelled ‘c’ are calibrated as is explained in the text. Those labelled ‘i’ are implied by other parameters. The remaining parameters are postulated a priori. The value for ρ_a follows from β, by noting that each model period represents 30 years.

Table 5.3: Allocation and welfare in the fertility model

	(a) ME	(b) SSO	(c) SWF $\beta_G = 0.4521$	(c) SWF $\beta_G = 0.5521$	(c) SWF $\beta_G = 0.6521$
$\hat{y}$	1.0000	1.6573	1.2003	1.3189	1.4197
$\hat{k}$	0.0720	0.3692	0.1222	0.1633	0.2076
$\hat{n}$	1.3478	0.9864	1.2190	1.1462	1.0896
$\hat{n}_a$	1.0000%	−0.0456%	0.6623%	0.4558%	0.2865 %
$\hat{r}$	3.3219	−0.0136	1.6963	1.0761	0.6709
$\hat{r}_a$	5.0000%	−0.0456%	3.3615%	2.4648%	1.7260%
$\hat{w}$	0.5978	0.9929	0.7240	0.7940	0.8521
$\hat{c}^y$	0.3812	0.8044	0.5070	0.5832	0.6489
$\hat{c}^o$	0.7184	0.5390	0.6868	0.6579	0.6288
$\hat{l}$	1.1710	1.3605	1.2293	1.2661	1.2972
$\hat{l}^y$	0.8000	0.8536	0.8191	0.8299	0.8383
$\hat{l}^o$	0.3710	0.5069	0.4102	0.4362	0.4589
$\hat{\Lambda}^y$	0.5847	0.7593	0.6396	0.6872	0.7215

Note Panel (a) summarizes the steady-state equilibrium for the unmanaged market economy. Panel (b) is the Samuelson social optimum. Panel (c) is the SWF social optimum.

Finally, we check if the model is in equilibrium. In the steady-state equilibrium, both the resource constraint and budget constraint must be satisfied:

$$\hat{y} - (\hat{n} + \delta - 1)\hat{k} = \hat{c}^y + \frac{\hat{c}^o}{\hat{n}}, \tag{5.2.35}$$

$$\hat{c}^y + \frac{\hat{c}^o}{1+\hat{r}} + z\hat{w}\hat{n} = \hat{w}\left[1 + \frac{\lambda}{1+\hat{r}}\right]. \tag{5.2.36}$$

The individual welfare has been estimated from steady-state values of individual consumption and fertility. The main features of the steady-state market equilibrium are reported in column (a) of Table 5.3.

5.3 Welfare Analysis

In this section we investigate two concepts of social optimization. The first concept is the oldest one due to Samuelson. It is later discussed and demonstrated further by Deardoff (1976) and Michel and Pestieau (1993). It simply maximizes the lifetime utility of a young individual in the steady state. The second welfare concept postulates a dynamically consistent social welfare function, where the social planner treats equally all individuals within but not across generations.

5.3.1 Samuelson Social Welfare

Samuelson (1975) and many other writers in this field maximize steady-state welfare of a representative young individual. To investigate the existence of an interior solution for optimal fertility n this is a useful first step. The social planner maximizes the objective function:

$$\Lambda^y \equiv V(n) + \left[\frac{1}{1+\beta}(c^y)^{1-1/\sigma} + \frac{\beta}{1+\beta}(c^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}, \tag{5.3.1}$$

subject to the resource constraints:

$$y + (1-\delta-n)k = c^y + \frac{c^o}{n}, \tag{5.3.2}$$

where $y = f(k,l)$ and $l = 1 - zn + \frac{\lambda}{n}$.

We use the insights of Michel and Pestieau (1993) for further investigation. Define maximum attainable resources conditional on n:

$$\Phi(n) \equiv \max_k f\left(k, 1 - zn + \frac{\lambda}{n}\right) - (n + \delta - 1)k. \tag{5.3.3}$$

The first-order condition for the optimization on the right-hand side turns out to be:

$$f_1\left(k, 1 - zn + \frac{\lambda}{n}\right) = n + \delta - 1. \tag{5.3.4}$$

Solve for the optimal k, we obtain:

$$k^*(n) = \begin{cases} \left(\frac{\alpha\Omega}{n+\delta-1}\right)^{1/(1-\alpha)} \left[1 - zn + \frac{\lambda}{n}\right], & \xi = 1 \\ \left(\frac{1}{1-\alpha}\right)^{\xi/(1-\xi)} \left[\left(\frac{\alpha\Omega}{n+\delta-1}\right)^{1-\xi} - \alpha\right]^{\xi/(1-\xi)} \left[1 - zn + \frac{\lambda}{n}\right]. & \xi \neq 1 \end{cases} \tag{5.3.5}$$

Hence the optimal resources are given by:

$$\Phi^*(n) = \Omega\left[\alpha k^*(n)^{1-1/\xi} + (1-\alpha)\left[1 - zn + \frac{\lambda}{n}\right]^{1-1/\xi}\right]^{1/(1-1/\xi)} - (n+\delta-1)k^*(n). \tag{5.3.6}$$

Michel and Pestieau (1993, P. 355) claim that $\Phi(n)$ is decreasing in n (for $z = \lambda = 1 - \delta = 0$).

Now let's define the indirect utility function as:

$$\Psi(n) \equiv V(n) + \Psi^c(n), \tag{5.3.7}$$

where $\Psi^c(n)$ denotes the indirect utility of the consumption part:

$$\Psi^c(n) \equiv \max_{c^y, c^o}\left[\frac{1}{1+\beta}(c^y)^{1-1/\sigma} + \frac{\beta}{1+\beta}(c^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}, \tag{5.3.8}$$

which subjects to:

$$c^y + \frac{c^o}{n} = \Phi(n). \tag{5.3.9}$$

The first-order conditions for the optimization of the indirect utility of the consumption part yields the following Euler equation:

$$c^o = (\beta n)^\sigma c^y, \tag{5.3.10}$$

From Equations (5.3.9) and (5.3.10) we obtain:

$$c^y = \frac{\Phi(n)}{1+\beta^\sigma n^{\sigma-1}}, \tag{5.3.11}$$

$$c^o = \frac{\beta^\sigma n^\sigma \Phi(n)}{1+\beta^\sigma n^{\sigma-1}}. \tag{5.3.12}$$

Substitute c^y and c^o into the indirect utility function we get:

$$\Psi^c(n) = \begin{cases} \Phi(n)(1+\beta)^{-1}(\beta n)^{\beta/(1+\beta)}, & \text{for } \sigma = 1 \\ \Phi(n)(1+\beta)^{-1}\left[\frac{1+\beta^\sigma n^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. & \text{for } \sigma \neq 1 \end{cases} \tag{5.3.13}$$

For an interior optimum we need $\Psi'(n^*) = 0$ and $\Psi''(n^*) < 0$ for some $n^* > 0$ subject to the constraint. The Lagrangian of the social planner's problem under Samuelson social optimum is given by:

$$\mathcal{L}^s = V(n) + \left[\frac{1}{1+\beta}(c^y)^{1-1/\sigma} + \frac{\beta}{1+\beta}(c^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)} + \mu\left[f(k, 1-zn+\frac{\lambda}{n}) + (1-\delta)k - c^y - \frac{c^o}{n} - nk\right]. \tag{5.3.14}$$

The first-order conditions for youth and old age consumption, number of children and capital intensity are given by:

$$\frac{\partial \mathcal{L}^s}{\partial c^y} = x\zeta(c^y)^{-1/\sigma} - \mu = 0, \tag{5.3.15}$$

$$\frac{\partial \mathcal{L}^s}{\partial c^o} = x(1-\zeta)(c^o)^{-1/\sigma} - \frac{\mu}{n} = 0, \tag{5.3.16}$$

$$\frac{\partial \mathcal{L}^s}{\partial n} = V'(n) - \mu\left[k + f_2\left(z + \frac{\lambda}{n^2}\right) - \frac{c^o}{n^2}\right] = 0, \tag{5.3.17}$$

$$\frac{\partial \mathcal{L}^s}{\partial k} = f_1(k,l) - (\delta + n - 1) = 0, \tag{5.3.18}$$

where x is defined as:

$$x \equiv \left[\zeta(c^y)^{1-1/\sigma} + (1-\zeta)(c^o)^{1-1/\sigma}\right]^{1/(\sigma-1)}. \tag{5.3.19}$$

From the Euler equation (5.3.10), x can be written as:

$$x = (c^y)^{1/\sigma}\left[\frac{1+\beta^\sigma n^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. \tag{5.3.20}$$

Thus, from Equation (5.3.15) μ is given by:

$$\mu = x\zeta(c^y)^{-1/\sigma} = \frac{1}{1+\beta}\left[\frac{1+\beta^\sigma n^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. \tag{5.3.21}$$

The optimal conditions characterizing the interior maximum are given by:

$$nk = f(k,l) + (1-\delta)k - \left[1+\beta^\sigma n^{\sigma-1}\right]c^y, \tag{5.3.22}$$

$$V'(n) = \mu\left[k + \left(z + \frac{\lambda}{n^2}\right) f_2(k,l) - \frac{(\beta n)^\sigma c^y}{n^2}\right], \tag{5.3.23}$$

$$f_1(k,l) = \delta + n - 1, \tag{5.3.24}$$

$$l = 1 - zn + \frac{\lambda}{n}, \tag{5.3.25}$$

which can be solved for n, k, c^y and l (x, μ and c^o follow readily).

To compare the Samuelson social welfare (SSO) with the market equilibrium (ME), we first write down the optimal conditions determining the number of children in both schemes. Under SSO we have:

$$\frac{V'(n)}{U_1(c^y, c^o)} = k + \left(z + \frac{\lambda}{n^2}\right) f_2(k,l) - \frac{c^o}{n^2}, \tag{5.3.26}$$

which is derived by substituting Equation (5.3.10) and (5.3.15) into (5.3.23)[7]. Under ME we have:

$$\frac{V'(n)}{U_1\left(c^y, c^o\right)} = zf_2(k, l), \tag{5.3.27}$$

where we have used Equation (5.2.12) and (5.2.24). On the left-hand side is the ratio between the marginal utility of an additional child and the marginal utility from an extra unit of consumpton; on the right-hand side is the relative cost of an additional child to consumption[8].

In the laissez-faire market equilibrium, individuals do not take into account the *capital dilution* effect, which is captured by the term k in (5.3.26) and increases the cost of an additional child. When the number of children increases, a higher capital stock is needed for them to be productive in the next period. Individuals ignore the fact that more children should have induced them to save more. Another factor that individuals do not take into account is the *intergenerational transfer* effect. More children means that there are more working individuals to support each retired individual. Therefore, the old-age labour supply becomes a smaller fraction of total labour supply (which is captured by $\frac{\lambda}{n^2} f_2(k, l)$) and a higher output in the future is shared with the same number of pensioners (which is captured by $-\frac{c^o}{n^2}$). Since old-age consumption (c^o) is higher than old-age income ($\lambda f_2(k, l)$), the *intergenerational transfer* effect reduces the cost of an additional child, $-\frac{c^o - \lambda f_2(k,l)}{n^2} < 0$. This is also verified in Table 5.3 ($c^o = 0.5390 > \lambda f_2(k, l) = \lambda w = 0.4965$). However, it is still unknown to us which effect dominates the other.

From Table 5.3, we find that the *capital dilution* effect dominates the *intergenerational transfer* effect, ($k = 0.3692 > \frac{c^o - \lambda f_2(k,l)}{n^2} = 0.0437$). As the cost of an additional child increases, the marginal utility of each child must rise, which leads parents to choose a smaller number of children: n is reduced from 1.3478 under ME to 0.9864 under SSO. Individuals save more for the next period (k increases from 0.0720 to 0.3692) and produce a higher output (y increases from 1.0000 to 1.6573) under SSO. They also enjoy a much higher youth consumption (c^y increases from 0.3812 to 0.8044) and slightly lower old-age consumption (c^o decreases from 0.7184 to 0.5390). And because the social planner internalizes the external effects of fertility, individuals enjoy a higher welfare level under SSO (Λ^y increases from 0.5847 to 0.7593).

5.3.2 Social Welfare Function

Social welfare function weights the generations in such a way that the social optimum is dynamically consistent. Specifically, as was stressed by Calvo and Obstfeld

[7] $U_1\left(c^y, c^o\right)$ is defined as:

$$U_1\left(c^y, c^o\right) \equiv \frac{\partial U\left(c^y, c^o\right)}{\partial c^y}.$$

[8] The marginal cost of consumption is 1.

(1988), it is imperative that the old generation in the planning period is treated appropriately by applying reverse discounting. Hence, the social welfare function is given by:

$$\begin{aligned}
SW_t &\equiv \frac{1}{\beta_G}\Lambda_{t-1}^y + \Lambda_t^y + \beta_G \Lambda_{t+1}^y + \ldots \\
&= \frac{1}{\beta_G}\left[V(n_{t-1}) + \left[\zeta \left(c_{t-1}^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_t^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}\right] \\
&\quad + V(n_t) + \left[\zeta \left(c_t^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_{t+1}^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)} \\
&\quad + \beta_G \left[V(n_{t+1}) + \left[\zeta \left(c_{t+1}^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_{t+2}^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}\right] + \ldots
\end{aligned} \tag{5.3.28}$$

where β_G is the social planner's discount factor ($0 < \beta_G < 1$). Recall that in Equation (5.2.5) β is defined as the private discount factor. Note that we apply a Mill-type social welfare function as in Van Groezen et al. (2003).

The economy-wide resource constraint is given by:

$$y_t + (1-\delta)\,k_t = c_t^y + \frac{c_t^o}{n_{t-1}} + n_t k_{t+1}. \tag{5.3.29}$$

The production function per young individual is defined as:

$$y_t = f(k_t, l_t) \equiv \Omega\left[\alpha k_t^{1-1/\xi} + (1-\alpha) l_t^{1-1/\xi}\right]^{1/(1-1/\xi)}, \tag{5.3.30}$$

where labour supply at time t is $l_t = 1 - z n_t + \frac{\lambda}{n_{t-1}}$.

At time t the social planner chooses time paths for c_τ^y, c_τ^o, n_τ and $k_{\tau+1}$ (for $\tau = t, t+1, ...$) in order to maximize SW_t subject to the resource constraint and the time-cost of children constraint. The initial conditions are k_t, c_{t-1}^y and n_{t-1}.

The Lagrangian of the social planner's problem is given by:

$$\begin{aligned}
\mathcal{L}_t^g &= \frac{1}{\beta_G}\left[V(n_{t-1}) + \left[\zeta \left(c_{t-1}^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_t^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}\right] \\
&+ V(n_t) + \left[\zeta \left(c_t^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_{t+1}^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)} \\
&+ \beta_G \left[V(n_{t+1}) + \left[\zeta \left(c_{t+1}^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_{t+2}^o\right)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}\right] + \ldots \\
&+ \mu_t \left[f\left(k_t, 1 - z n_t + \frac{\lambda}{n_{t-1}}\right) + (1-\delta)\,k_t - c_t^y - \frac{c_t^o}{n_{t-1}} - n_t k_{t+1}\right] \\
&+ \beta_G \mu_{t+1} \left[f\left(k_{t+1}, 1 - z n_{t+1} + \frac{\lambda}{n_t}\right) + (1-\delta)\,k_{t+1} - c_{t+1}^y - \frac{c_{t+1}^o}{n_t} - n_{t+1} k_{t+2}\right] + \ldots
\end{aligned} \tag{5.3.31}$$

The first-order conditions for youth and old age consumption, number of children and future capital intensity in period t are given by:[9]

$$\frac{\partial \mathcal{L}_t^g}{\partial c_t^y} = \zeta x_t \left(c_t^y\right)^{-1/\sigma} - \mu_t = 0, \tag{5.3.32}$$

$$\frac{\partial \mathcal{L}_t^g}{\partial c_t^o} = \frac{1-\zeta}{\beta_G} x_{t-1} \left(c_t^o\right)^{-1/\sigma} - \frac{\mu_t}{n_{t-1}} = 0, \tag{5.3.33}$$

$$\frac{\partial \mathcal{L}_t^g}{\partial n_t} = V'(n_t) + \beta_G \mu_{t+1} \frac{c_{t+1}^o - \lambda f_2(k_{t+1}, l_{t+1})}{n_t^2} - \mu_t \left[z f_2(k_t, l_t) + k_{t+1}\right] = 0, \tag{5.3.34}$$

$$\frac{\partial \mathcal{L}_t^g}{\partial k_{t+1}} = -\mu_t n_t + \beta_G \mu_{t+1} \left[f_1(k_{t+1}, l_{t+1}) + 1 - \delta\right] = 0, \tag{5.3.35}$$

where x_t is defined as:

$$x_t \equiv \left[\zeta \left(c_t^y\right)^{1-1/\sigma} + (1-\zeta)\left(c_{t+1}^o\right)^{1-1/\sigma}\right]^{1/(\sigma-1)}. \tag{5.3.36}$$

Thus we can derive the optimality conditions for the first-best social optimum (FBSO):

$$n_t k_{t+1} = f(k_t, l_t) + (1-\delta) k_t - c_t^y - \frac{c_t^o}{n_{t-1}}, \tag{5.3.37}$$

$$\mu_t = \zeta x_t \left(c_t^y\right)^{-1/\sigma} = \frac{1-\zeta}{\beta_G} n_{t-1} x_{t-1} \left(c_t^o\right)^{-1/\sigma}, \tag{5.3.38}$$

$$V'(n_t) = \mu_t \left[z f_2(k_t, l_t) + k_{t+1}\right] + \beta_G \mu_{t+1} \frac{\lambda f_2(k_{t+1}, l_{t+1}) - c_{t+1}^o}{n_t^2}, \tag{5.3.39}$$

$$\mu_t n_t = \beta_G \mu_{t+1} \left[f_1(k_{t+1}, l_{t+1}) + 1 - \delta\right], \tag{5.3.40}$$

$$l_t = 1 - z n_t + \frac{\lambda}{n_{t-1}}. \tag{5.3.41}$$

In the steady-state FBSO we have $k_{t+1} = k_t = \hat{k}$, $n_{t+1} = n_t = \hat{n}$, $c_t^y = c_{t+1}^y = \hat{c}^y$, $c_t^o = c_{t+1}^o = \hat{c}^o$, and $\mu_{t+1} = \mu_t = \hat{\mu}$, so that the consumption Euler equation is followed by:

$$\frac{\hat{c}^o}{\hat{c}^y} = \left(\frac{\beta \hat{n}}{\beta_G}\right)^{\sigma}, \tag{5.3.42}$$

and it follows that:

$$\hat{x} = \left(\hat{c}^y\right)^{1/\sigma} \left[\frac{1 + \beta^\sigma \beta_G^{1-\sigma} \hat{n}^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}, \tag{5.3.43}$$

$$= \frac{\beta_G}{\beta \hat{n}} \left(\hat{c}^o\right)^{1/\sigma} \left[\frac{1 + \beta^\sigma \beta_G^{1-\sigma} \hat{n}^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. \tag{5.3.44}$$

$$\hat{\mu} = \frac{1}{1+\beta} \left[\frac{1 + \beta^\sigma \beta_G^{1-\sigma} \hat{n}^{\sigma-1}}{1+\beta}\right]^{1/(\sigma-1)}. \tag{5.3.45}$$

[9] The first-order conditions for other periods $\tau = t+1, t+2, \ldots$ are similar.

The optimal conditions characterizing the interior maximum are:

$$\hat{n}\hat{k} = f(\hat{k},\hat{l}) + (1-\delta)\,\hat{k} - \left[1 + \left(\frac{\beta}{\beta_G}\right)^{\sigma} \hat{n}^{\sigma-1}\right] \hat{c}^y, \tag{5.3.46}$$

$$V'(\hat{n}) = \hat{\mu}\left[zf_2(\hat{k},\hat{l}) + \hat{k}\right] + \beta_G\hat{\mu}\frac{\lambda f_2(\hat{k},\hat{l}) - \left(\frac{\beta\hat{n}}{\beta_G}\right)^{\sigma}\hat{c}^y}{\hat{n}^2}, \tag{5.3.47}$$

$$\hat{n} = \beta_G\left[f_1(\hat{k},\hat{l}) + 1 - \delta\right], \tag{5.3.48}$$

$$\hat{l} = 1 - z\hat{n} + \frac{\lambda}{\hat{n}}, \tag{5.3.49}$$

which can be solved for $\hat{n}$, $\hat{k}$, $\hat{c}^y$, and $\hat{l}$ ($\hat{x}$, $\hat{\mu}$ and $\hat{c}^o$ follow readily).

When we compare the Social welfare function (SWF) equilibrium with the market equilibrium (ME) and Samuelson social welfare (SSO) equilibrium, it is convenient to write down the optimal conditions concerning the number of children. In the steady-state FBSO we have:

$$\frac{V'(n)}{U_1(c^y,c^o)} = k + zf_2(k,l) + \beta_G\left(\frac{\lambda f_2(k,l)}{n^2} - \frac{c^o}{n^2}\right), \tag{5.3.50}$$

where we have used Equations (5.3.38), (5.3.42) and (5.3.47). Comparing the optimality condition (5.3.50) with condition (5.3.27), we find that in the laissez-faire market equilibrium (ME), individuals do not consider the *capital dilution* effect, which is captured by the term k. Each additional child needs exactly the same per capita capital to keep the per capita production at the former level when they join the labour force. Another factor that individuals fail to internalize is the *intergenerational transfer* effect, which is captured by the term in the bracket in (5.3.50). As more children are born, there will be abundant labour force in the next period, thus old-age labour force will be less important (which is captured by $\frac{\lambda f_2(k,l)}{n^2}$), and more production will be shared with each retired individual (which is captured by $\frac{c^o}{n^2}$). The *capital dilution* effect increases the cost of each additional child ($k = 0.1633 > 0$), and the *intergenerational transfer* effect reduces the cost of each additional child ($-\frac{c^o - \lambda f_2(k,l)}{n^2} < 0$), which is verified by Table 5.3 ($c^o = 0.6579 > \lambda f_2(k,l) = \lambda w = 0.3970$). The *intergenerational transfer* effect is also discounted by the social discount factor $\beta_G = 0.5521$.

Once we substitute the fertility rate $n = 1.1462$ under SWF in Table 5.3 into the *intergenerational transfer* effect, we find that the *capital dilution* effect dominates the *intergenerational transfer* effect ($k = 0.1633 > \beta_G\frac{c^o - \lambda f_2(k,l)}{n^2} = 0.1096$), the cost of an additional child rises. The marginal utility from an extra child increases with the cost of an additional child. Therefore, under SWF, parents choose a smaller number of children compared to ME: n is reduced from 1.3478 to 1.1462. A higher capital stock is saved for the next period under SWF (k increases from 0.0720 to 0.1633), which leads to a higher output level (y increases from 1.0000 to 1.3189). Moreover, individuals enjoy a higher youth consumption (c^y increases from 0.3812 to 0.5832) and a slightly lower old-age consumption (c^o decreases from 0.7184 to 0.6579). They also enjoy a higher welfare level under SWF (Λ^y increases from 0.5847 to 0.6872) since the social planner internalizes the *capital dilution* effect and the *intergenerational transfer* effect.

To note the differences between the SWF equilibrium and the SSO equilibrium, we examine conditions (5.3.50) and (5.3.26). We find that under SWF the *intergenerational transfer* effect is discounted by the social discount factor β_G. Because the output in the future has a smaller weight (discounted by the social planner) in the utility function, individuals decide to save less, which leads to a smaller capital stock (k reduces from 0.3692 to 0.1633) and smaller total output (y reduces from 1.6573 to 1.3189). Because of a smaller capital stock under SWF, the overall external effect of fertility ($k + \beta_G \left(\frac{\lambda f_2(k,l)}{n^2} - \frac{c^o}{n^2} \right) = 0.0537$) is much smaller than in SSO ($k + \frac{\lambda f_2(k,l)}{n^2} - \frac{c^o}{n^2} = 0.3255$), so that the price of an additional child declines. The lower cost of each extra child leads to a higher fertility rate in SWF (n increases from 0.9864 to 1.1462). Comparing to SSO, individuals in SWF consume less in youth (c^y declines from 0.8044 to 0.5832) and slightly more in old age (c^o increases from 0.5390 to 0.6579). They also enjoy a slightly lower welfare level in SWF as the social planner discounts future utilities (Λ^y decreases from 0.7593 to 0.6872).

We prefer SWF over SSO as the objective function for social welfare analysis. The SSO focuses on the steady state while SWF considers all generations and is dynamically consistent. Since the SWF function is dynamic, it provides us the tools to analyze individual decisions (transfers, savings, etc.) across generations. Furthermore, the SWF function puts a smaller weight on future consumption and income, which is also consistent across time. In the following analysis we are going to take the SWF equilibrium as the first best social optimum (FBSO).

5.4 Child taxes and lump-sum transfers

5.4.1 Market Economy with Lump-sum Taxes

We notice that the resource constraints are exactly the same under market economy (ME) and social welfare function (SWF) economy. The output and remaining capital are either consumed in the current period or to be saved as future capital to produce in the next period. If the government could design a tax scheme such that individuals are induced to choose the optimal number of children and golden-rule savings rate, the first best social optimum (FBSO) can be replicated from the market economy (ME) with government taxes.

Assume that individual utility function has not been changed:

$$\Lambda_t^y \equiv V(n_t) + U(c_t^y, c_{t+1}^o), \tag{5.4.1}$$

The government taxes $\psi_t w_t$ for each child, and redistributes the tax as a pay-roll subsidy $w_t \eta_t$ to each individual. An individual receives a subsidy T_t^y in youth and has to pay a lump-sum tax T_{t+1}^o to the government in old age. The government has a balanced budget:

$$\psi_t w_t n_t = w_t \eta_t, \tag{5.4.2}$$

$$T_t^y + \frac{T_{t+1}^o}{n_t} = 0, \tag{5.4.3}$$

The individual budget constraint can now be written as:

$$c_t^y + s_t + \psi_t w_t n_t = w_t(1 - z n_t + \eta_t) + T_t^y, \tag{5.4.4}$$

$$c_{t+1}^o = (1 + r_{t+1}) s_t + \lambda w_{t+1} + T_{t+1}^o, \tag{5.4.5}$$

The consolidated budget constraint can be written as:

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} + \psi_t w_t n_t = w_t(1 - z n_t + \eta_t) + T_t^y + \frac{\lambda w_{t+1} + T_{t+1}^o}{1 + r_{t+1}}, \tag{5.4.6}$$

Thus the Lagrangian is given by:

$$\begin{aligned} \mathcal{L}_t^y \equiv\ & V(n_t) + \left[\zeta (c_t^y)^{1-1/\sigma} + (1-\zeta)(c_{t+1}^o)^{1-1/\sigma}\right]^{1/(1-1/\sigma)} \\ & + \lambda_t^y \left[w_t(1 - z n_t + \eta_t) + T_t^y + \frac{\lambda w_{t+1} + T_{t+1}^o}{1 + r_{t+1}} - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} - \psi_t w_t n_t \right], \end{aligned} \tag{5.4.7}$$

We obtain the first-order conditions:

$$\frac{\partial \mathcal{L}_t^y}{\partial n_t} = V'(n_t) - \lambda_t^y (z + \psi_t) w_t = 0, \tag{5.4.8}$$

$$\frac{\partial \mathcal{L}_t^y}{\partial c_t^y} = \frac{\zeta x_t}{\left(c_t^y\right)^{1/\sigma}} - \lambda_t^y = 0, \tag{5.4.9}$$

$$\frac{\partial \mathcal{L}_t^y}{\partial c_{t+1}^o} = \frac{(1-\zeta) x_t}{\left(c_{t+1}^o\right)^{1/\sigma}} - \frac{\lambda_t^y}{1 + r_{t+1}} = 0, \tag{5.4.10}$$

where x_t is defined as:

$$x_t \equiv \left[\zeta (c_t^y)^{1-1/\sigma} + (1-\zeta)(c_{t+1}^o)^{1-1/\sigma}\right]^{1/(\sigma-1)}. \tag{5.4.11}$$

From the first-order conditions we obtain optimality conditions:

$$\frac{c_{t+1}^o}{c_t^y} = \left[\beta(1 + r_{t+1})\right]^{\sigma}, \tag{5.4.12}$$

$$\frac{V'(n_t)\left(c_t^y\right)^{1/\sigma}}{\zeta x_t} = (z + \psi_t) w_t. \tag{5.4.13}$$

Combining the Euler equation and the lifetime budget constraint we get:

$$c_t^y = \frac{1}{1+\beta^\sigma(1+r_{t+1})^{\sigma-1}}\left[w_t(1-zn_t)+T_t^y+\frac{\lambda w_{t+1}+T_{t+1}^o}{1+r_{t+1}}\right], \tag{5.4.14}$$

$$\begin{aligned} s_t = n_t k_{t+1} = {} & \frac{\beta^\sigma(1+r_{t+1})^{\sigma-1}}{1+\beta^\sigma(1+r_{t+1})^{\sigma-1}}\left[w_t(1-zn_t)+T_t^y\right] \\ & - \frac{1}{1+\beta^\sigma(1+r_{t+1})^{\sigma-1}}\left[\frac{\lambda w_{t+1}+T_{t+1}^o}{1+r_{t+1}}\right], \end{aligned} \tag{5.4.15}$$

$$c_{t+1}^o = \left[\beta(1+r_{t+1})\right]^\sigma c_t^y. \tag{5.4.16}$$

From Eqs. (5.4.13) and (5.4.15) we find that the intergenerational tax T^y has been designed to induce individuals to choose the golden-rule savings rate. When individuals decide to save more or less than the social optimal, the government can adjust the tax T^y to lead them to save at the social optimal. The child tax ψ has been designed to induce individuals to have the optimal number of children. Interestingly, we find that these two taxes work in tandem with each other. Neither tax could work without the complement of the other. Together they pin down the optimal capital stock and fertility rate for society.

5.4.2 Steady-state Decentralization

Now we are going to decentralize the stead-state first best social optimal (BFSO) economy from the market economy with the help of government taxes. We will show that, with appropriate design of government taxes, the managed market economy will reach the first best social optimum.

Recall from the social welfare function (SWF) economy:

$$\mu_t n_t = \beta_G \mu_{t+1}\left[f_1(k_{t+1}, l_{t+1}) + 1 - \delta\right], \tag{5.4.17}$$

In the steady-state $\mu_t = \mu^*$ for all t, so we have:

$$\hat{n} = \beta_G\left[f_1(\hat{k},\hat{l}) + 1 - \delta\right] = \beta_G(1+\hat{r}), \tag{5.4.18}$$

where $\hat{k}$ is the socially optimal capital stock per young worker. To replicate the SWF equilibrium, we first use the lump-sum transfer T_t^y to make sure that individual savings will increase. Consequently, the capital stock k_t will increase and finally converge to $k_t = \hat{k}$. Meanwhile we use the child tax ψ_t to induce individuals to have the socially optimal number of children $\hat{n}$. Recall from the market economy (ME) with government taxes:

$$\begin{aligned} n_t k_{t+1} = {} & \frac{\beta^\sigma(1+r_{t+1})^{\sigma-1}}{1+\beta^\sigma(1+r_{t+1})^{\sigma-1}}\left[w_t(1-zn_t)+T_t^y\right] \\ & - \frac{1}{1+\beta^\sigma(1+r_{t+1})^{\sigma-1}}\left[\frac{\lambda w_{t+1}+T_{t+1}^o}{1+r_{t+1}}\right], \end{aligned} \tag{5.4.19}$$

$$\frac{V'(n_t)}{U_1\left(c_t^y, c_{t+1}^o\right)} = (z+\psi_t) f_2(k_t, l_t), \tag{5.4.20}$$

where we have used equations (5.4.13) – (5.4.15) . In the steady state we obtain:

$$\tilde{n}\tilde{k} = \frac{\beta^\sigma(1+\tilde{r})^{\sigma-1}}{1+\beta^\sigma(1+\tilde{r})^{\sigma-1}}\left[\tilde{w}(1-z\tilde{n})+\tilde{T}^y_t\right] - \frac{1}{1+\beta^\sigma(1+\tilde{r})^{\sigma-1}}\left[\frac{\lambda\tilde{w}+\tilde{T}^o_{t+1}}{1+\tilde{r}}\right], \tag{5.4.21}$$

$$\frac{V'(\tilde{n})}{U_1\left(\tilde{c}^y,\tilde{c}^o\right)} = (z+\psi_t)f_2(\tilde{k},\tilde{l}), \tag{5.4.22}$$

The planner sets the capital stock per young worker and fertility rate equal to $\hat{k}$ and $\hat{n}$, respectively. Then we will have $\tilde{k}=\hat{k}$, $\tilde{r}=\hat{r}$, $\tilde{w}=\hat{w}$, $\tilde{n}=\hat{n}$ and $\tilde{l}=\hat{l}$. Taking into account of government budget constraint, there must exist a lump-sum tax on (transfer to) the young $\tilde{T}^y$ and a child tax $\tilde{\psi}$ such that the right capital stock $\hat{k}$ and fertility rate $\hat{n}$ are chosen in the market economy:

$$\hat{n}\hat{k} = \frac{\beta^\sigma(1+\hat{r})^{\sigma-1}}{1+\beta^\sigma(1+\hat{r})^{\sigma-1}}\left[\hat{w}(1-z\hat{n})+\tilde{T}^y\right] - \frac{1}{1+\beta^\sigma(1+\hat{r})^{\sigma-1}}\left[\frac{\lambda\hat{w}-\hat{n}\tilde{T}^y}{1+\hat{r}}\right], \tag{5.4.23}$$

$$\frac{V'(\hat{n})}{U_1(\hat{c}^y,\hat{c}^o)} = (z+\tilde{\psi})f_2(\hat{k},\hat{l}) = \hat{k}+zf_2(\hat{k},\hat{l})+\beta_G\left(\frac{\lambda f_2(\hat{k},\hat{l})-\hat{c}^o}{\hat{n}^2}\right), \tag{5.4.24}$$

Where the $\tilde{T}^y$ and $\tilde{\psi}$ are the policy instruments. Rearranging the equations, we find:

$$\tilde{T}^y = -\frac{\tilde{T}^o}{\hat{n}} = \frac{(1+\beta)\hat{n}\hat{k}+\left[\frac{\lambda\beta_G}{\hat{n}}-\beta(1-z\hat{n})\right]\hat{w}}{\beta+\beta_G}, \tag{5.4.25}$$

$$\tilde{\psi} = \frac{\hat{k}+\beta_G\left(\frac{\lambda f_2(\hat{k},\hat{l})-\hat{c}^o}{\hat{n}^2}\right)}{f_2(\hat{k},\hat{l})}. \tag{5.4.26}$$

We can also verify that, in this managed market equilibrium, the consumption will be the same as in the SWF equilibrium. In the steady state, the capital market clears, $\hat{s}=\hat{n}\hat{k}$. Combining Eq. (5.4.4) and (5.4.5), we obtain:

$$\begin{aligned}\tilde{c}^y+\frac{\tilde{c}^o}{\hat{n}} &= \hat{w}(1-z\hat{n})-\hat{s}+\frac{1+r}{\hat{n}}\hat{s}+\frac{\lambda}{\hat{n}}\hat{w},\\ &= \hat{w}\hat{l}+(1+\hat{r}-\hat{n})\hat{k},\\ &= y(\hat{k},\hat{l})+(1-\hat{n}-\delta)\hat{k}.\end{aligned} \tag{5.4.27}$$

Substituting in Eq. (5.4.18), the Euler equation in the steady state can be written as:

$$\frac{\tilde{c}^o}{\tilde{c}^y} = \left(\frac{\beta\hat{n}}{\beta_G}\right)^\sigma. \tag{5.4.28}$$

Since the resource constraint and the Euler equation are exactly the same as in the SWF equilibrium, the consumption in the managed market equilibrium must be kept the same.

We have shown that the managed market economy with two appropriate tax policies could reach the first best social optimum (FBSO). This is because the taxes have led individuals to take into account the externalities of fertility: the *capital dilution* effect and the *intergenerational transfer* effect. By designing a lump-sum tax that transfers resources from the old to the young, the government has induced individuals to save more, which internalizes the *capital dilution* effect. By putting a tax on each child, the government has led individuals to reconsider the marginal social cost of children. In our simulations, we find that the *capital dilution* effect more than offsets the *intergenerational transfer* effect, which implies raising an extra child is more costly for the society. When fertility rate is endogenous rather than exogenous, the fertility choice and savings decision interact with each other. These two optimal choices are determined simultaneously once the two tax polices have been implemented.

5.4.3 Transitional Dynamics

We have shown that, with appropriate design of government taxes, the managed market economy will reach the socially optimal equilibrium in the steady state. Now we ask the following question: is it better to move the market economy to the SWF equilibrium with the nudge of government taxes? If we compare the ME equilibrium and SWF equilibrium in the steady state in Table (5.3), the answer is probably yes. In the SWF equilibrium we have higher output, higher capital stock, higher wage rate, higher labor participation, and more importantly, higher welfare level (in terms of lifetime utility) . This is mainly because the fertility rate has become lower: the social externalities of fertility choices have been internalized by the family with the incentives of government taxes.

But the steady-state comparison only reveals us a corner of the picture. We also care about the welfare of generations of people who are affected by the tax policy. To find out wether this tax policy is Pareto improving for all generations we study the transitional dynamic effects of introducing the government taxes. Substituting in the numbers of the SWF equilibrium we can calculate the values of the policy instruments:

$$\tilde{T}^y = -\frac{\tilde{T}^o}{1+n} = \frac{(1+\beta)\hat{n}\hat{k} + \left[\frac{\lambda\beta_G}{\hat{n}} - \beta(1-z\hat{n})\right]\hat{w}}{\beta+\beta_G} = 0.1115, \tag{5.4.29}$$

$$\tilde{\psi} = \frac{\hat{k} + \beta_G\left(\frac{\lambda f_2(\hat{k},\hat{l}) - \hat{c}^o}{\hat{n}^2}\right)}{f_2(\hat{k},\hat{l})} = 0.0676. \tag{5.4.30}$$

Substitute the values of $\tilde{T}^y$ and $\tilde{\psi}$ into the managed market equilibrium model as listed in Table (5.4). At shock-time $t = 0$, individuals face a child tax ψ which are

Table 5.4: Managed Market equilibrium in the fertility model

$$
\begin{aligned}
c^o_{t+1} &= c^y_t \left[\beta(1+r_{t+1})\right]^{\sigma} \\
c^o_t &= \lambda w_t + T^o_t + (1+r_t) n_t k_t \\
s_t &= n_t k_{t+1} \\
s_t &= \frac{\beta^{\sigma}(1+r_{t+1})^{\sigma-1}}{1+\beta^{\sigma}(1+r_{t+1})^{\sigma-1}} \left[w_t(1-z n_t) + T^y_t\right] \\
&\quad - \frac{1}{1+\beta^{\sigma}(1+r_{t+1})^{\sigma-1}} \left[\frac{\lambda w_{t+1} + T^o_{t+1}}{1+r_{t+1}}\right] \\
n_t &= n_{\min} + \left(\frac{\gamma(1+\beta)}{(z+\psi) w_t}\right)^{1/(1-\theta)} \left[\frac{1+\beta^{\sigma}(1+r_{t+1})^{\sigma-1}}{1+\beta}\right]^{1/[(\theta-1)(\sigma-1)]} \\
w_t &= (1-\alpha)\,\Omega^{(\xi-1)/\xi} \left(\frac{y_t}{l_t}\right)^{1/\xi} \\
r_t + \delta &= \alpha \Omega^{(\xi-1)/\xi} \left(\frac{y_t}{k_t}\right)^{1/\xi} \\
0 &= T^y_t + \frac{T^o_t}{n_t} \\
y_t &= \Omega \left[\alpha k_t^{1-1/\xi} + (1-\alpha) l_t^{1-1/\xi}\right]^{1/(1-1/\xi)} \\
l_t &= 1 - z n_t + \frac{\lambda}{n_{t-1}}
\end{aligned}
$$

Note Variables: youth consumption c^y_t, (planned) old-age consumption c^o_{t+1}, saving of the young s_t, current capital stock k_t, fertility rate n_t, wage rate w_t, real interest rate r_t, tax on the old T^o_t, output y_t, labour supply l_t.

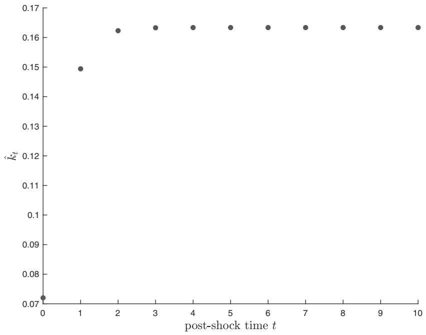
---

redistributed to the family as a payroll subsidy, and each young individual receives a government subsidy T^y which are financed by taxing the old people T^o. Using Dynare, I compute the transition path from the unmanaged market equilibrium (ME) to the first best social optimum (FBSO). In the steady state, as expected, the managed market economy reaches the FBSO.

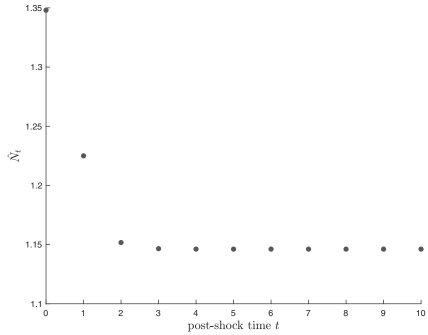
Figure (5.2) depicts some key features of the transition process. Panel (a) shows that the capital per young worker starts at $\tilde{k} = 0.072$ and converges to the FBSO value of $\hat{k} = 0.1633$ in about 4 to 5 periods (120-150 years). Panel (b) shows that the child tax induces each parent to have less children. The fertility rate declines from $\tilde{n} = 1.3480$ to $\hat{n} = 1.1462$. Panel (c) and (d) show that youth consumption increases from $\tilde{c}^y = 0.3812$ to $\hat{c}^y = 0.5832$ and old-age consumption decreases from $\tilde{c}^o = 0.7184$ to $\hat{c}^o = 0.6579$. This is mainly due to the fact that the government

Figure 5.2: Transition from ME to FBSO

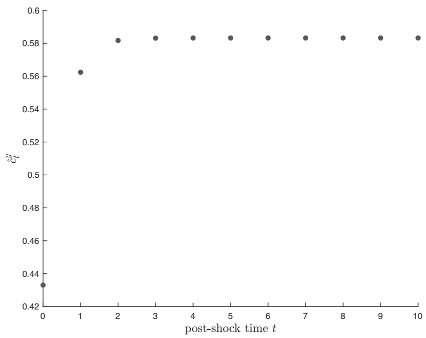
(a) Capital $\hat{k}_t$



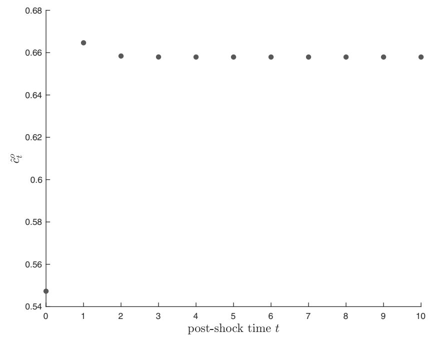
(b) Fertility $\hat{n}_t$



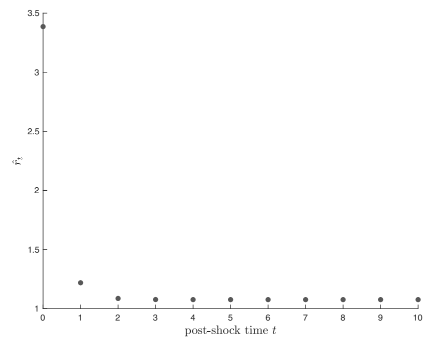
(c) Youth consumption $\hat{c}_t^y$



(d) Ol-age consumption c_t^o



(e) Interest rate $\hat{r}_t$



(f) Wage rate $\hat{w}_t$

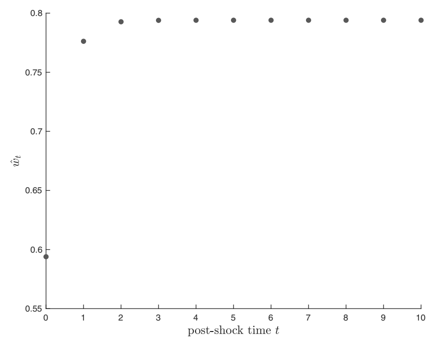
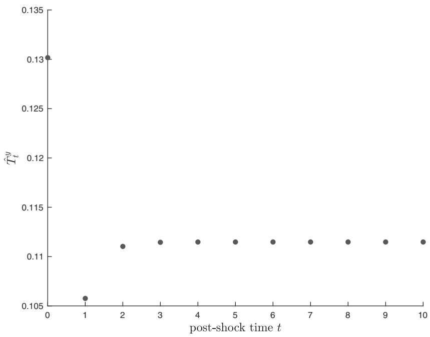
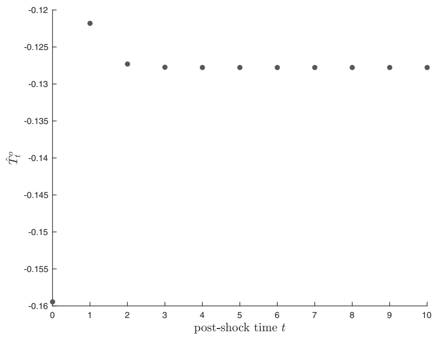


Figure 5.3: Lump-sum transfers and individual welfare

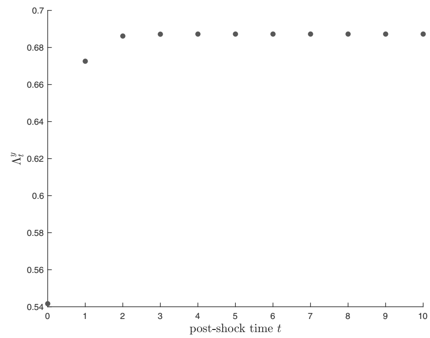
(a) Transfers to the young $\hat{T}_t^y$



(b) Transfers to the old $\hat{T}_t^o$



(c) Individual welfare Λ_t^y



transfers resources from the old to the young, leading young people to consume and save more than before. Panel (e) and (f) show the subsequent changes in interest rate and wage rate, which follows the change in capital stock. Interest rate declines from $\tilde{r} = 3.3218$ to $\hat{r} = 1.0761$ and wage rate rises from $\tilde{w} = 0.5978$ to $\hat{w} = 0.7940$.

Figure (5.3) shows us the lump-sum transfers needed to decentralize the FBSO. The government transfers resources from the old to the young. In the steady state the young is subsidized by $\hat{T}_t^y = 0.1115$ and the old is taxed by $\hat{T}_t^o = 0.1278$. It also illustrates the transition process of individual welfare: it first decrease from $\tilde{\Lambda}_t^y = 0.5847$ to $\tilde{\Lambda}_t^y = 0.5416$, then increases monotonically to $\hat{\Lambda}_t^y = 0.6872$. This is because the first generation—the old at the time of the shock—are taxed without any compensation. Hence we arrive at the conclusion that the first generation is going to suffer from the transition from the unmanaged market equilibrium to the first best social optimum, while in the long run all other generations will benefit from the transition. In the steady state, the social externalities of fertility – the *capital dilution* effect and *intergenerational transfer* effect – have been fully internalized.

5.5 Conclusion

In this Chapter we have developed a two-period overlapping generations model which features endogenous fertility and CES utility and production functions. Consumers are assumed to derive direct utility from their offspring, and are free to choose the number of children they want subject to their budget constraints. Following Michel and Pestieau (1993), we find that there exists an interior solution for the optimal fertility rate, given CES utility and production functions with low elasticity of substitution. By a plausibly calibrated version of the model, we show that the unmanaged market economy does *not* generally coincide with the first best social optimum. In the laissez faire market economy, individuals do not take into account the social externalities of fertility: when the fertility rate is too high, it requires a larger fraction of output to be inserted into the capital formation to maintain the capital-labour ratio, which is dubbed the '*capital dilution*' effect; when the fertility rate is too low, there will be too few young workers to support the old generation, which is called the '*intergenerational transfer*' effect.

We solved the social planner's problem, which internalized the social externalities of fertility. We compared two welfare concepts: the Samuelson Social Welfare (SSO) concept and Social Welfare Function (SWF) concept. The SSO focuses on the steady state generation whilst SWF considers all generations and discounts the utility of future generations with a social discount factor β_G. We prefer the SWF as the objective function for the social welfare analysis as it provides us the convenience to analyze individual choices across generations and gives us a better picture of the transitional dynamics. In the SWF equilibrium (FBSO), individual welfare has been enhanced due to the internalization of externalities of fertility.

The FBSO can be decentralized from the managed market economy with appropriate design of government tax and transfers. With a child tax and intergenerational lump-sum transfers, the market economy can replicate the FBSO. The intuition behind is simple: the government tax and transfers have led individuals to take into account the externalities of fertility. The golden-rule savings rate and the optimal fertility rate are pinned down simultaneously once the government tax and transfers have been introduced. More importantly, the introduction of government tax and transfers yields a higher individual welfare level not only in the steady state, but all generations after the shock will benefit from the policy as shown by the transitional dynamics analysis. The transition from the laissez faire market economy to the fist best social optimum is an improvement for future generations in the long run.

In this Chapter we have built and calibrated a model that verifies Samuelson's Serendipity theorem. We extend the work of Michel and Pestieau (1993) by introducing endogenous fertility decisions and a dynamically consistent Social Welfare Function. We show that there exists an interior optimal fertility rate, which the unmanaged market economy does not pick up automatically, but can be reached from a managed market economy with the nudge of government tax and transfers. Samuelson's '*most-golden golden-rule state*' can not only be reached by serendipity, it can also be reached by the right government policies.

Chapter 6

Conclusion

This thesis is a collection of micro-founded small macroeconomic models (and conjugate micro models) attempting to shed light on the optimal financial and social arrangements for ageing.[1] Specifically, we extended and enriched the Samuelson (1958) - Diamond (1965) model by including lifetime uncertainty, asymmetric information, heterogenous individuals and endogenous fertility decisions to study the insurance of longevity risk, optimal fertility rates for society and the macroeconomic effects of private and public annuities in/without presence of bequests. Within a minimal theoretical framework we carefully relax some stylized but unrealistic assumptions behind the original overlapping-generations models (for instance, exogenous fertility assumption) and explore the welfare implications of private and public old-age insurance arrangements with the aid of numerical simulations. The advantage of this method is two-fold: the macroeconomic effects of a financial instrument can be computed immediately and exactly by computers, which could hardly be shown analytically; while the minimal theoretical framework provide us with strong and persuading intuitions. It is our intention to avoid the cluttering of large complex models, as they barely provide any meaningful insights to our understanding of the problem of ageing.[2]

In the second chapter we built a two-period overlapping generations model to study the macroeconomic effects of private and public old-age insurance with asymmetric information. With full information of mortality (or lifetime certainty), Yaari (1965) has proved the benefit of annuities as a financial instrument to insure against the longevity risk. Annuity, as an old-age insurance instrument, is a financial contract that pays a premium upon survival into the old age and nothing upon death. In real world private annuity markets are usually plagued by adverse selection: healthy individuals often over-invest while unhealthy individuals are crowded out of the market. The adverse selection is aggravated by the health-wealth nexus, as healthy individuals are more likely to be wealthy. Thus we assumed that individuals differ by *two dimensions* of heterogeneity: health and earning ability.[3] With heterogeneous individuals and asymmetric information, our findings are very different from that of Yarri (1965). Firstly, we find that private annuity market with asymmetric information reduces the output per efficiency unit of labour and capital intensity compared with full information. Furthermore, the introduction of social security (public annuity) aggravates the adverse selection in the private annuity market and reduces the output per efficiency unit of labour and capital intensity further. Due to the redistribution role of the social security, the individual welfare effects of social security depends both on the individual type and pension benefit rule. Absent the redistribution from unhealthy to healthy types, if the pension benefits are proportional to contributions, then social security makes everybody worse off in the long run. Last but not least, our simulation has shown that privatizing social security is not Pareto

[1]For the strengths and limitations of small macroeconomic models, see Turnovsky (2011).

[2]I do not mean to obscure the meaning of large complex models. Its strength lies in approximating the real-world economy and is often used by banks and government research institutions.

[3]Our model is built on Heijdra and Reijnders (2012), which assumed one dimension of heterogeneity: health.

improving for all generations: the higher welfare equilibrium in the long run comes at the price of the shock time healthy individuals.

In the third chapter we used a two-period life-cycle model developed by Abel (1986) to show that *annuity puzzle* can easily be explained by the interplay of asymmetric information and bequest motives in the rational domain. *Annuity puzzle* usually refers to the fact that few people voluntarily purchase annuities while theoretically they are very attractive financial instruments since their returns are superior than non-annuitized savings. Davidoff *et al.* (2005) has proved that neither asymmetric information nor bequest motives alone could explain the almost non-existent private annuity market. We combined the two commonly considered reasons for the *annuity puzzle* and found that together they would cause the low-health individuals to leave the annuity market. The intuition behind the 'interplay mechanism' is that asymmetric information and adverse selection would cut the premium of annuities and bequest motives would enhance the value of non-annuitized assets. If the private annuity is priced too dear due to adverse selection and individuals have strong bequest motives, they would opt out of the annuity market. Later we extended the model by including a pay-as-you-go social security system and introducing a correlation between health and earning ability. The former one, as a public insurance program, would crowd out the private annuities and the latter would aggravate the adverse selection on the annuity market as heavier investment of healthier individuals pushes down the annuity premium ever further.

In the fourth chapter we examine the macroeconomic effects of opening up an annuity market in the presence of bequest motives. It is generally believed that individuals derive utilities from purchasing annuities if they don't have a bequest motive. As shown in chapter 3, individuals still benefit from purchasing actuarially-fair - or at least not too unfair - annuities when they have a bequest motive. But the private benefit from purchasing annuities does not guarantee a social welfare improvement. Heijdra *et al.* (2014) have shown that in a dynamically efficient economy, opening up an annuity market reduces unintended bequests and the intergenerational transfer from the old to the young, and moves the economy away from the Golden Rule. As a consequence, it reduces the capital intensity and individual welfare. In that paper they have proved the *Tragedy of Annuitization* without the presence of intended bequest motives. We revisit the question by assuming intended bequest motives such that a utility value is attached to intended bequests and the bequest motive is operative. We were able to show that a stronger bequest motive is associated with higher capital accumulation as more assets are (intentionally) transferred from the old to the young. More importantly, we find that when bequest motives are accounted for, opening up an annuity market will again lead to a decrease in capital accumulation and a drop in individual welfare. *Tragedy of Annuitization* prevails in the presence of bequest motives, though the negative general equilibrium effect is dampened by the bequest motives. Furthermore, having an imperfect annuity market seems to restrain individuals from annuitizing their assets and diminished the negative general equilibrium effects of opening up an annuity market. An extremely thin, almost non-existent annuity market is not necessarily a bad thing after all.

In the fifth chapter we developed a two-period overlapping generations model to study the optimal fertility decisions for the society. We assume endogenous fertility decisions such that individuals not only derive utility from their consumption but also from their offsprings. Starting from Michel and Pestieau (1993) we used CES utility/technology functions and a numerical method to find the interior solution for the social optimum fertility decision. We investigated two concepts of social optimization: the Samuelson social welfare concept with maximize the steady-state welfare of a representative young individual, and Social welfare function where the social optimum is dynamically consistent. The former criteria is static and the latter one is preferred. Later we showed that a managed market economy with child taxes and intergenerational transfers is able to replicate the first-best social optimum (FBSO) under the social welfare function criteria. Not only do individuals enjoy a higher welfare level in the first-best social optimum, we also showed that the transitional path is an improvement for all generations except for the old generation at the shock period.

Our thesis has focused on the problem of ageing and by combination of theoretical and numerical methods we were able to show that to insure against the longevity risk, individual decisions are very different from macroeconomic optimal choices. Private annuities may be attractive to some healthy individuals but opening up an annuity market is by no means beneficial to the society. The value of annuities (annuity market) should be examined in a general equilibrium framework, in tandem with bequest motives. We also explored the optimal fertility for the society, which generally does not coincide with individual choices in *laissez faire*. With Child taxes and intergenerational transfers, however, the government may induce individuals to choose the optimal quantity and quality of children the society needs. After all, with the prevalence of public pension programs, what provides a better insurance to the old generation than a well-educated right-sized population of the next generation?

Chapter 7

Nederlandse Samenvatting

Dit proefschrift is een verzameling van microgefundeerde, kleine macro-economische en geconjugeerde micromodellen, die samen pogen licht te werpen op de optimale financiële en sociale regelingen voor ouder worden[1]. Specifiek hebben wij het model van Samuelson (1958) - Diamond (1965) uitgebreid met de volgende factoren: levenslooponzekerheid, asymmetrische informatie, heterogene individuen en endogene vruchtbaarheids-beslissingen, dit om levensduurrisico, optimale maatschappelijke vruchtbaarheidscijfers en de macro-economische effecten van private en publieke lijfrentes, mét en zonder erfenismotief, te bestuderen. Binnen een minimaal theoretisch kader versoepelen we op zorgvuldige wijze enkele gestileerde maar onrealistische aannames achter het oorspronkelijke model van de overlappende generaties (zoals de veronderstelling van exogene invloeden op vruchtbaarheid). Daarnaast gebruiken we getalsmatige simulaties om te onderzoeken welke implicaties private en publieke oudedagsverzekeringen hebben voor de welvaart. Deze methode biedt twee voordelen: we kunnen de macro-economische effecten van een financieel instrument onmiddellijk en met behulp van computers precies berekenen, iets dat op analytische wijze nauwelijks mogelijk zou zijn. Daarnaast levert het minimale theoretisch kader sterke en overtuigende intuïties op. Wij willen de 'rommeligheid' van grote, complexe modellen vermijden omdat ze nauwelijks zinvolle inzichten opleveren in het probleem van het ouder worden[2].

Het tweede hoofdstuk omvat het ontwerp van een tweeperiodenmodel van overlappende generaties om de macro-economische effecten te onderzoeken van private en publieke oudedagsverzekeringen met asymmetrische informatie. Yaari (1965) heeft het voordeel aangetoond van lijfrentes als financieel instrument tegen het levensduurrisico bij volledige informatie over sterfte (oftewel: levensduurzekerheid). Als verzekeringsinstrument voor de oude dag is lijfrente een financieel contract dat een premie uitkeert bij ouderdom en niets uitbetaalt bij overlijden. In werkelijkheid worden private lijfrentemarkten doorgaans geteisterd door negatieve selectie: gezonde mensen investeren vaak teveel, terwijl ongezonde mensen uit de markt verdrongen worden. Die negatieve selectie wordt nog versterkt door de *health-wealth nexus*, het verband tussen gezondheid en welvaart, dat laat zien dat gezonde mensen een grotere kans hebben ook rijk te zijn. We hebben dan ook aangenomen dat individuen verschillen vertonen op twee dimensies van heterogeniteit: gezondheid en

[1] Zie Turnovsky (2011) voor de sterke punten en beperkingen van kleine macro-economische modellen.
[2] Ik wil het belang van grote, complexe modellen niet bagatelliseren. Hun kracht ligt erin dat ze de werkelijk economie benaderen; ze worden dus vaak gebruikt door banken en overheidsonderzoekinstituten.

verdiencapaciteit[3]. Uitgaande van heterogene personen en asymmetrische informatie, verschillen onze resultaten drastisch van die van Yaari (1965). Ten eerste stellen we vast dat private lijfrentemarkten met asymmetrische informatie minder output per arbeidsefficiëntie-eenheid en minder kapitaalintensiteit opleveren dan in een situatie met volledige informatie. Daarnaast versterkt de invoering van sociale zekerheid (publieke lijfrente) de negatieve selectie in de private lijfrentemarkt en zorgt zij voor een verdere vermindering van de output per arbeidsefficiëntie-eenheid en van de kapitaalintensiteit. Door de herverdeling die inherent is aan de sociale zekerheid verschillen de effecten op de individuele welvaart per persoon, afhankelijk van het type pensioenregeling. Als er geen herverdeling plaatsvindt van ongezonde naar gezonde personen en als de pensioenuitkeringen in verhouding staan tot de premies, dan is iedereen uiteindelijk slechter af met sociale zekerheid. Ten slotte toont onze simulatie aan dat privatisering van de sociale zekerheid niet voor alle generaties leidt tot een Paretoverbetering: het hogere welvaartsevenwicht op de lange termijn gaat ten koste van gezonde personen ten tijde van de schok.

In hoofdstuk 3 tonen we met behulp van een levenscyclusmodel met twee perioden, ontwikkeld door Abel (1986), aan dat de oplossing van de *annuity puzzle - de lijfrentepuzzel* - ligt in een samenspel van asymmetrische informatie en een erfenismotief in het rationele domein. De term *annuity puzzle* verwijst doorgaans naar het feit dat maar weinig mensen vrijwillig lijfrentes aanschaffen, terwijl het in theorie zeer aantrekkelijke financiële instrumenten zijn, omdat de rendementen hoger liggen dan die van andere, niet op lijfrente gebaseerde spaarvormen. Davidoff *et al.* hebben aangetoond dat asymmetrische informatie noch het erfenismotief op zich voldoende verklaring zijn voor het vrijwel ontbreken van een private lijfrentemarkt. Door deze twee algemeen genoemde redenen voor de lijfrentepuzzel te combineren, hebben we ontdekt dat ze er samen voor zorgen dat mensen met een slechte gezondheid de lijfrentemarkt verlaten. Onze intuïtie over dit ‘samenspel’ zegt ons dat informatie-asymmetrie en negatieve selectie de lijfrentepremie zouden drukken en dat het erfenismotief de waarde van niet-annuïtairgemaakte bezittingen zou opdrijven. Als een private lijfrente te duur is door negatieve selectie, zouden mensen met een sterk erfenismotief ervoor kunnen kiezen de lijfrentemarkt te verlaten. We hebben het model vervolgens uitgebreid met een *pay-as-you-go*-socialezekerheidsstelsel en met een correlatie tussen gezondheid en verdiencapaciteit. Het eerste is een publieke verzekering en zou de private lijfrentes uit de markt drukken. De laatstgenoemde correlatie zou de negatieve selectie op de

[3] Ons model bouwt voort op dat van Heijdra en Reijnders (2012), dat uitging van één dimensie van heterogeniteit: gezondheid.

lijfrentemarkt versterken: grotere investeringen door gezondere mensen zorgen voor nog meer premieverlaging.

In hoofdstuk 4 onderzoeken we de macro-economische effecten van de opening van een lijfrentemarkt in aanwezigheid van erfenismotieven. Algemeen wordt aangenomen dat mensen nut ontlenen aan de aanschaf van lijfrentes als ze geen erfenismotief hebben. Zoals aangetoond in hoofdstuk 3, kunnen mensen echter nog steeds baat hebben bij het kopen van actuarieel eerlijke – of in ieder geval niet al te oneerlijke – lijfrentes, zelfs als er sprake is van een erfenismotief. Het particuliere voordeel van het aanschaffen van lijfrentes vormt echter geen garantie voor een verbetering van de sociale zekerheid. Heijdra *et al.* (2014) hebben laten zien dat het openen van een lijfrentemarkt in een dynamisch-efficiënte economie leidt tot een afname van het aantal onbedoelde nalatenschappen en van (waarde)overdracht tussen de generaties – van oud naar jong – en dat de economie erdoor verwijderd raakt van de gouden balansregel. Als gevolg hiervan daalt de kapitaalintensiteit en de individuele welvaart. In hun paper hebben zij de tragiek van lijfrentes, de *Tragedy of Annuitization*, aangetoond in afwezigheid van gewenste erfenismotieven. Wij komen terug op deze kwestie door uit te gaan van gewenste erfenismotieven, waardoor gewenste erfenismotieven een nutswaarde krijgen en het erfenismotief actief is. Wij hebben kunnen aantonen dat een sterker erfenismotief verbonden is met hogere vermogensopbouw, aangezien meer activa (bewust) worden overgedragen van ouderen naar jongeren. Nog belangrijker is onze vaststelling dat het openen van een lijfrentemarkt ook zal leiden tot lagere vermogensopbouw en lagere individuele welvaart wanneer erfenismotieven worden meegenomen. De *Tragedy of Annuitization* blijft overeind wanneer er sprake is van erfenismotieven, al temperen deze motieven het negatieve effect op het algemene evenwicht. Daarbij weerhoudt een imperfecte lijfrentemarkt mensen ervan hun activa om te zetten in lijfrentes en vermindert deze de negatieve effecten die het openen van een lijfrentemarkt heeft op het algemeen evenwicht. Een zeer dunne, haast onbestaande lijfrentemarkt is dus niet per se slecht.

In hoofdstuk 5 hebben we een tweeperiodenmodel van overlappende generaties ontwikkeld om de optimale vruchtbaarheidsbeslissingen voor een samenleving te bestuderen. We gaan uit van endogene vruchtbaarheidsbeslissingen, waarbij individuen niet alleen nut ontlenen aan hun (eigen) consumptie maar ook aan (die van) hun nageslacht. Voortbouwend op Michel en Pestieau (1993), maken we gebruik van CES-nuts- / technologische functies en een numerieke methode om tot de optimale interne vruchtbaarheidsbeslissing te komen voor een samenleving. We hebben twee concepten van sociale optimalisering onderzocht: het sociale welvaartsconcept van Samuelson, dat de stabiele staat van welzijn van een representatief jong persoon maximaliseert; en de sociale welvaartsfunctie, waarin het sociaal optimum dynamisch consistent is. Het eerstgenoemde criterium is statisch en het laatstgenoemde geniet de voorkeur. Vervolgens hebben we aangetoond dat een gestuurde markteconomie met kinderbelastingen en (waarde)overdracht tussen de generaties in staat is het first-best sociaal optimum (FBSO) na te bootsen volgens de criteria van de sociale welvaartsfunctie. Individuen profiteren niet alleen *in* een first-best sociaal optimum van een hoger welvaartsniveau, we hebben ook aangetoond dat *het pad ernaartoe* Pareto-optimaal is.

Dit proefschrift richt zich op het probleem van het ouder worden. Door theoretische en numerieke methoden te combineren, hebben wij kunnen aantonen dat individuele beslissingen over het verzekeren van het levensduurrisico sterk verschillen van macro-economische optimale keuzes. Private lijfrentes kunnen aantrekkelijk zijn voor bepaalde gezonde personen, maar het openen van een lijfrentemarkt komt de samenleving absoluut niet ten goede. De waarde van lijfrentes (de lijfrentemarkt) zou nader moeten worden onderzocht in het kader van een algemeen evenwicht, in combinatie met erfenismotieven. We hebben ook gekeken naar de optimale vruchtbaarheidsgraad voor de samenleving, iets dat doorgaans niet samengaat met individuele keuzes als onderdeel van een laisser-faire-houding. Met behulp van kinderbelastingen en (waarde)overdracht tussen de generaties kunnen overheden individuen er echter toe aanzetten het aantal en het soort kinderen te kiezen dat voor de samenleving optimaal is. Immers, wat zou, gezien de dominantie van publieke pensioenstelsels, een betere verzekering voor de oudere generatie vormen dan een hoogopgeleide nieuwe generatie van de juiste omvang?

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